A simplified ordinal analysis of first-order reflection

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Abstract

In this note we give a simplified ordinal analysis of first-order reflection. An ordinal notation system OT is introduced based on ψ-functions, and a wellfoundedness proof of it is done in terms of distinguished classes. Provable Σ_1-sentences on $L_{\omega_1^{CK}}$ are bounded through cut-elimination on operator controlled derivations.

1 Introduction

Let ORD denote the class of all ordinals, $A \subset ORD$ and α a limit ordinal. α is said to be Π_n*-reflecting* on A iff for any Π_n-formula $\phi(x)$ and any $b \in L_\alpha$, if $\langle L_\alpha, \in\rangle \models \phi(b)$, then there exists a $\beta \in A \cap \alpha$ such that $b \in L_\beta$ and $\langle L_\beta, \in\rangle \models \phi(b)$. Let us write

$$\alpha \in rM_n(A) :\Leftrightarrow \alpha \text{ is } \Pi_n\text{-reflecting on } A.$$

Also α is said to be Π_n*-reflecting* iff α is Π_n-reflecting on ORD.

It is not hard to show that the assumption that the universe is Π_n-reflecting is proof-theoretically reducible to iteratability of the lower operation rM_{n-1} (and Mostowski collapsings), cf. [7].

In this paper we aim an ordinal analysis of Π_n-reflection. Though such an analysis was done by Pohlers and Stegert [12] using reflection configurations introduced in M. Rathjen [14], and in [3,4,9] with the complicated combinatorial arguments of ordinal diagrams and finite proof figures, our approach is simpler in view of combinatorial arguments. In [3, 4, 9], our ramification process is akin to a tower, i.e., has an exponential structure. Mahlo classes $Mh_k(\xi)$ defined in Definition 2.3 to resolve or approximate Π_N-reflection are based on similar structure, but here we avoid the complicated combinatorial arguments with the help of operator controlled derivations introduced by W. Buchholz [11].

On the other side our wellfoundedness proof is based on distinguished classes introduced by W. Buchholz [10], and similar to our proof in [1, 3, 4].

Our theorems run as follows. Let $\mathsf{KP}\Pi_N$ denote the set theory for Π_N-reflecting universes, $\mathsf{KP}\omega$ the Kripke-Platek set theory with the axiom of infinity, and $\mathsf{KP}\ell$ a set theory for limits of admissibles. OT is a computable notation system of ordinals defined in section 3, $\Omega = \omega_1^{CK}$ and ψ_Ω is a collapsing function such that $\psi_\Omega(\alpha) < \Omega$. $\mathbb{K}$ is an ordinal term denoting the least Π_N-reflecting ordinal in the theorems.

Theorem 1.1 *Suppose* $\mathsf{KP}\Pi_N \vdash \theta$ *for a* $\Sigma_1(\Omega)$*-sentence* θ*. Then we can find an* $n < \omega$ *such that for* $\alpha = \psi_\Omega(\omega_n(\mathbb{K}+1))$, $L_\alpha \models \theta$.

Theorem 1.2 $\mathsf{KP}\Pi_N$ *proves that* each *initial segment* $\{\alpha \in OT : \alpha < \psi_\Omega(\omega_n(\mathbb{K}+1))\}$ $(n = 1, 2, \ldots)$ *is well-founded.*

Thus we obtain the following Theorem 1.3.

Theorem 1.3

$$\psi_\Omega(\varepsilon_{\mathbb{K}+1}) = |\mathsf{KP}\Pi_N|_{\Sigma_1^\Omega} := \min\{\alpha \leq \omega_1^{CK} : \forall\theta \in \Sigma_1(\mathsf{KP}\Pi_N \vdash \theta^{L_\Omega} \Rightarrow L_\alpha \models \theta)\}.$$

Let us mention the contents of this paper. In the next section 2 we define simultaneously iterated Skolem hulls $\mathcal{H}_\alpha(X)$ of sets X of ordinals, ordinals $\psi_\kappa^{\vec{\xi}}(\alpha)$ for regular cardinals κ, $\alpha < \varepsilon_{\mathbb{K}+1}$ and sequences $\vec{\xi} = (\xi_2, \ldots, \xi_{N-1})$ of ordinals $\xi_i < \varepsilon_{\mathbb{K}+2}$, and classes $Mh_k^\alpha(\xi)$ under the *assumption* that a Π^1_{N-2}-indescribable cardinal $\mathbb{K}$ exists. It is shown that for $2 \leq k < N$, $\alpha < \varepsilon_{\mathbb{K}+1}$ and each $\xi < \varepsilon_{\mathbb{K}+2}$, ($\mathbb{K}$ is a Π^1_{N-2}-indescribable cardinal) $\to \mathbb{K} \in Mh_k^\alpha(\xi)$ in $\mathsf{ZF} + (V = L)$.

In section 3 a computable notation system OT of ordinals is extracted. In section 4 following [11], operator controlled derivations for $\mathsf{KP}\Pi_N$ is introduced, and inference rules for Π_N-reflection are eliminated from derivations in section 5. This completes an upper bound Theorem 1.1.

In the second part of this note, we show a lower bound Theorem 1.2. After a preliminary, rudimentary facts on distinguished sets are stated in section 6. Since many properties of distingusihed classes are seen as in [2, 4], we will give only a sketch of a proof in many cases. However we dealt with ordinal diagrams in [2, 4], and here with ordinal terms based on ψ-functions. Although these two seem to be closely related each other, we give a full proof of some facts for readers' conveniences. In section 7 we define a tower relation on ordinal terms largely as in [3, 4]. We need to take pains in subsection 7.1 to embed collapsing relations on ordinal terms into an exponential structure as for ordinal diagrams. In the final section 8 our wellfoundedness proof is concluded, and a corollary on conservative extensions is obtained in the end of the note.

Ω_α denotes the continuous closure of the α-th admissible ordinal for $\alpha > 0$. This means that $\Omega_1 = \omega_1^{CK}$ and $\Omega_\omega = \sup\{\Omega_n : 0 < n < \omega\} = \sup\{\tau_n : n < \omega\}$, where τ_α denotes the α-th admissible ordinal. Let $X < \alpha :\Leftrightarrow \forall\beta \in X(\beta < \alpha)$, $\alpha \leq X :\Leftrightarrow \exists\beta \in X(\alpha \leq \beta)$ and $X \leq Y :\Leftrightarrow \forall\alpha \in X\exists\beta \in Y(\alpha \leq \beta)$.

IH denotes the Induction Hypothesis, MIH the Main IH and SIH the Subsidiary IH. We are assuming tacitly the axiom of constructibility $V = L$. Throughout of this note $N \geq 3$ is a fixed integer.

2 Ordinals for Π_N-reflection

In this section we work in the set theory

$$\mathsf{ZFLK}_N := \mathsf{ZFL} + (\exists\mathbb{K}(\mathbb{K} \text{ is } \Pi^1_{N-2}\text{-indescribable}))$$

where $\mathsf{ZFL} = \mathsf{ZF} + (V = L)$ and $N \geq 3$ is a fixed integer.

Let $ORD \subset V$ denote the class of ordinals, $\mathbb{K}$ the least Π^1_{N-2}-indescribable cardinal, and Reg the set of regular ordinals below $\mathbb{K}$. Θ denotes finite sets of ordinals$\leq \mathbb{K}$.

Let $ORD^\varepsilon \subset V$ and $<^\varepsilon$ be Δ-predicates such that for any transitive and wellfounded model V of $\mathsf{KP}\omega$, $<^\varepsilon$ is a well ordering of type $\varepsilon_{\mathbb{K}+1}$ on ORD^ε for the order type $\mathbb{K}$ of the class ORD in V.

$u, v, w, x, y, z, \ldots$ range over sets in the universe, $a, b, c, \alpha, \beta, \gamma, \ldots$ range over ordinals$< \Lambda := \varepsilon_{\mathbb{K}+1}$, $\xi, \zeta, \nu, \mu, \iota, \ldots$ range over ordinals$< \varepsilon_{\mathbb{K}+2}$, $\vec{\xi}, \vec{\zeta}, \vec{\nu}, \vec{\mu}, \vec{\iota}, \ldots$ range over finite sequences over ordinals$< \varepsilon_{\mathbb{K}+2}$, and $\pi, \kappa, \rho, \sigma, \tau, \lambda, \ldots$ range over regular ordinals. θ, δ denote formulas.

Define simultaneously classes $\mathcal{H}_\alpha(X)$, classes $Mh^\alpha_k(\xi)$, and ordinals $\psi^{\vec{\xi}}_\kappa(\alpha)$ as follows. We see that these are Σ_1-definable as a fixed point in ZFL, cf. Proposition 2.5.

Definition 2.1 *1.* Let $\vec{\xi} = (\xi_0, \ldots, \xi_{m-1})$ be a sequence of ordinals.

(a) *length* $lh(\vec{\xi}) := m$ and *i-th component* $\vec{\xi}_i := \xi_i$ for $i \leq lh(\vec{\xi})$.

(b) The set of components $K(\vec{\xi}) := \{\vec{\xi}_i : i < lh(\vec{\xi})\} = \{\xi_0, \ldots, \xi_{m-1}\}$.

(c) Sequences consisting of a single element (ξ) is identified with the ordinal ξ, and $\emptyset$ denotes the *empty sequence.* $\vec{0}$ denotes ambiguously a zero-sequence, $\forall i < lh(\vec{0})(\vec{0}_i = 0)$ with its length $0 \leq lh(\vec{0}) \leq N-1$.

(d) $\vec{\xi} * \vec{\mu} = (\xi_0, \ldots, \xi_{m-1}) * (\mu_0, \ldots, \nu_{n-1}) = (\xi_0, \ldots, \xi_{m-1}, \mu_0, \ldots, \mu_{n-1})$ denotes the *concatenated* sequence of $\vec{\xi}$ and $\vec{\mu}$.

2. $\Lambda = \varepsilon_{\mathbb{K}+1}$ denotes the next epsilon number above the least Π_{N-2}-indescribable cardinal $\mathbb{K}$, and $\varepsilon_{\mathbb{K}+2}$ the next epsilon number above Λ.

For $i < \omega$ and $\xi < \varepsilon_{\mathbb{K}+2}$, $\Lambda_i(\xi)$ is defined recursively by $\Lambda_0(\xi) = \xi$ and $\Lambda_{i+1}(\xi) = \Lambda^{\Lambda_i(\xi)}$.

3. For a non-zero ordinal $\xi < \varepsilon_{\mathbb{K}+2}$, its Cantor normal form with base Λ is uniquely determined

$$\xi =_{NF} \sum_{i \leq m} \Lambda^{\xi_i} a_i = \Lambda^{\xi_m} a_m + \cdots + \Lambda^{\xi_0} a_0 \tag{1}$$

where $\xi_m > \cdots > \xi_0$, $0 < a_i < \Lambda$.

(a) $K(\xi) = \{a_i : i \leq m\} \cup \bigcup\{K(\xi_i) : i \leq m\}$ is the set of *components* of ξ with $K(0) = \emptyset$.

For a sequence $\vec{\xi} = (\xi_0, \ldots, \xi_{n-1})$ of ordinals $\xi_i < \varepsilon_{\mathbb{K}+2}$, $K^2(\vec{\xi}) := \bigcup\{K(\xi) : \xi \in K(\vec{\xi})\} = \bigcup\{K(\xi_i) : i < n\}$.

(b) For $\xi > 1$, $te(\xi) = \xi_0$ in (1) is the *tail exponent*, and $he(\xi) = \xi_m$ is the *head exponent* of ξ, resp.
$Hd(\xi) := \Lambda^{\xi_m} a_m$, and $Tl(\xi) := \Lambda^{\xi_0} a_0$.
Put $te(i) := he(i) := Hd(i) := Tl(i) := i$ for $i \in \{0, 1\}$.

(c) $he^{(i)}(\xi)$ is the *i-th head exponent* of ξ, defined recursively by $he^{(0)}(\xi) = \xi$, $he^{(i+1)}(\xi) = he(he^{(i)}(\xi))$.
The *i-th tail exponent* $te^{(i)}(\xi)$ is defined similarly.

(d) $\zeta \leq_{pt} \xi$ designates that $\zeta =_{NF} \sum_{i \geq n} \Lambda^{\xi_i} a_i = \Lambda^{\xi_m} a_m + \cdots + \Lambda^{\xi_n} a_n$ for an $n\, (0 \leq n < m+1)$. $\zeta <_{pt} \xi :\Leftrightarrow \zeta \leq_{pt} \xi \,\&\, \zeta \neq \xi$.

4. For $A \subset ORD$, limit ordinals α and $i \geq 0$

$$\alpha \in M_{2+i}(A) :\Leftrightarrow A \cap \alpha \text{ is } \Pi^1_i\text{-indescribable in } \alpha$$

5. κ^+ denotes the next regular ordinal above κ.

6. $\Omega_\alpha := \omega_\alpha$ for $\alpha > 0$, $\Omega_0 := 0$, and $\Omega = \Omega_1$.

Proposition 2.2 $\xi < \mu \Rightarrow he(\xi) \leq he(\mu)$.

Let $a < \Lambda$, and φ denote the binary Veblen function. $\mathcal{H}_a(X)$ is the Skolem hull of $\{0, \mathbb{K}\} \cup X$ under the functions $+, \alpha \mapsto \omega^\alpha, (\alpha, \beta) \mapsto \varphi\alpha\beta\, (\alpha, \beta < \mathbb{K}), \alpha \mapsto \Omega_\alpha\, (\alpha < \mathbb{K}), (\kappa, \gamma) \mapsto \psi_\kappa\gamma\, (\gamma < a, \kappa \in Reg \cup \{\mathbb{K}\}), (\kappa, \vec{\nu}, \gamma) \mapsto \psi^{\vec{\nu}}_\kappa(\gamma)$ where $\max K^2(\vec{\nu}) \leq \gamma < a, \kappa \in Reg \cup \{\mathbb{K}\}$.

Definition 2.3 *1.*

$$\mathcal{H}_a[Y](X) := \mathcal{H}_a(Y \cup X)$$

for sets $Y \subset \mathbb{K}$.

2. Let for sequences $\vec{\nu} = (\nu_2, \ldots, \nu_{n-1})\, (n > 0)$,

$$\begin{array}{rcl} \vec{\nu} <_{tl} \xi & :\Leftrightarrow & \exists \vec{\mu} = (\mu_0, \ldots, \mu_{n-1})[\forall i \leq n-1(\nu_i < \mu_i) \\ & \& & \mu_0 \leq_{pt} \xi \,\&\, \forall i < n-1(\mu_{i+1} \leq_{pt} te(\mu_i))] \end{array} \tag{2}$$

3. (Inductive definition of $\mathcal{H}_a(X)$).

(a) $\{0, \mathbb{K}\} \cup X \subset \mathcal{H}_a(X)$.

(b) $x, y \in \mathcal{H}_a(X) \Rightarrow x + y \in \mathcal{H}_a(X)$, $x \in \mathcal{H}_a(X) \Rightarrow \omega^x \in \mathcal{H}_a(X)$, and $x, y \in \mathcal{H}_a(X) \cap \mathbb{K} \Rightarrow \varphi xy \in \mathcal{H}_a(X)$.

(c) $\mathbb{K} > \alpha \in \mathcal{H}_a(X) \Rightarrow \Omega_\alpha \in \mathcal{H}_a(X)$.

(d) If $\{b, \kappa\} \cup K^2(\vec{\nu}) \subset \mathcal{H}_a(X)$ with $\max K^2(\vec{\nu}) \leq b < a$ and $lh(\vec{\nu}) = N - 2$, then $\psi^{\vec{\nu}}_\kappa(b) \in \mathcal{H}_a(X)$.

4. (Definitions of $Mh^a_k(\xi)$ and $Mh^a_k(\vec{\xi})$) First let

$$\mathbb{K} \in Mh^a_N(0) :\Leftrightarrow \mathbb{K} \in M_N \Leftrightarrow \mathbb{K} \text{ is } \Pi_{N-2}\text{-indescribable.}$$

The classes $Mh^a_k(\xi)$ are defined for $2 \leq k < N$, and ordinals $a < \Lambda$, $\xi < \varepsilon_{\mathbb{K}+2}$. Let π be a regular ordinal$\leq \mathbb{K}$. Then for $\xi > 0$

$$\begin{array}{l} \pi \in Mh^a_k(\xi) :\Leftrightarrow \{a\} \cup K(\xi) \subset \mathcal{H}_a(\pi)\,\& \\ \forall \vec{\nu} <_{tl} \xi(K^2(\vec{\nu}) \subset \mathcal{H}_a(\pi) \Rightarrow \pi \in M_k(Mh^a_k(\vec{\nu}))) \end{array} \tag{3}$$

where $\vec{\nu} = (\nu_0, \ldots, \nu_{n-1})\,(0 < n \leq N-k)$ varies *non-empty* sequences of ordinals. For sequences $\vec{\nu}$

$$\pi \in Mh^a_k(\vec{\nu}) :\Leftrightarrow \pi \in \bigcap_{i<n} Mh^a_{k+i}(\nu_i).$$

By convention, let for $2 \leq k < N$,

$$\pi \in Mh^a_k(0) :\Leftrightarrow \pi \in Mh^a_2(\emptyset) :\Leftrightarrow \pi \text{ is a limit ordinal}$$

Note that by letting $\vec{\nu} = (0)$ for $\xi > 0$, $\pi \in Mh^a_k(\xi) \Rightarrow \pi \in M_k$. Also $\vec{0} <_{tl} 1$, and $Mh^a_k(1) = M_k$ since $te(1) = 1$.

5. (Definition of $\psi^{\vec{\xi}}_\kappa(a)$) Let $a < \Lambda$ be an ordinal, κ a regular ordinal and $\vec{\xi}$ a sequence of ordinals$< \varepsilon_{\mathbb{K}+2}$ such that $lh(\vec{\xi}) = N-2$, $\max K^2(\vec{\xi}) \leq a$, and $K^2(\vec{\xi}) \cup \{\kappa, a\} \subset \mathcal{H}_a(\kappa)$. Suppose $\kappa \in M_2(Mh^a_2(\vec{\xi}))$. Then let

$$\psi^{\vec{\xi}}_\kappa(a) := \min(\{\kappa\} \cup \{\pi \in Mh^a_2(\vec{\xi}) \cap \kappa : \mathcal{H}_a(\pi) \cap \kappa \subset \pi, K^2(\vec{\xi}) \cup \{\kappa, a\} \subset \mathcal{H}_a(\pi)\}) \tag{4}$$

Let

$$\psi_\kappa a := \psi^{\vec{0}}_\kappa a$$

where $lh(\vec{0}) = N-2$, $Mh^a_2(\vec{0}) = Lim$, and $\kappa \in M_2$, i.e., κ is a regular ordinal.

Proposition 2.4 *$b + c \in \mathcal{H}_a[\Theta](d) \Rightarrow c \in \mathcal{H}_a[\Theta](d)$, and $\omega^c \in \mathcal{H}_a[\Theta](d) \Rightarrow c \in \mathcal{H}_a[\Theta](d)$.*

The following Proposition 2.5 is easy to see.

Proposition 2.5 *Each of $x = \mathcal{H}_a(y)\,(a < \varepsilon_{\mathbb{K}+1}, y < \mathbb{K})$, $x = \psi_\kappa a$, $x \in Mh^a_k(\xi)$ and $x = \psi^{\vec{\xi}}_\kappa(a)$, is a Σ_1-predicate as fixed points in* ZFL.

Proof. This is seen from the facts that there exists a universal Π^1_n-formula, and by using it, $\alpha \in M_n(x)$ iff $\langle L_\alpha, \in\rangle \models m_n(x \cap L_\alpha)$ for some Π^1_{n+1}-formula $m_n(R)$ with a unary predicate R. □

Let $A(a)$ denote the conjunction of $\forall u < \mathbb{K}\exists! x[x = \mathcal{H}_a(u)]$, and $\forall \vec{\xi} \forall x(\max K^2(\vec{\xi}) \leq a\,\&\,K^2(\vec{\xi}) \cup \{\kappa, a\} \subset x = \mathcal{H}_a(\kappa) \to \exists! b \leq \kappa(b = \psi^{\vec{\xi}}_\kappa(a)))$,

where $lh(\vec{\xi}) = N-2$.

Since the cardinality of the set $\mathcal{H}_{\varepsilon_{\mathbb{K}+1}}(\pi)$ is π for any infinite cardinal $\pi \leq \mathbb{K}$, pick an injection $f : \mathcal{H}_{\varepsilon_{\mathbb{K}+1}}(\mathbb{K}) \to \mathbb{K}$ so that $f\text{"}\mathcal{H}_{\varepsilon_{\mathbb{K}+1}}(\pi) \subset \pi$ for any weakly inaccessibles $\pi \leq \mathbb{K}$.

Lemma 2.6 *1. $\forall a < \varepsilon_{\mathbb{K}+1}\, A(a)$.*

2. *$\pi \in Mh_k^a(\xi)$ is a Π^1_{k-1}-class on L_π uniformly for weakly inaccessible cardinals $\pi \leq \mathbb{K}$ and a, ξ. This means that for each k there exists a Π^1_{k-1}-formula $mh_k^a(x)$ such that $\pi \in Mh_k^a(\xi)$ iff $L_\pi \models mh_k^a(\xi)$ for any weakly inaccessible cardinals $\pi \leq \mathbb{K}$ with $f\text{"}(\{a\} \cup K(\xi)) \subset L_\pi$.*

3. *$\mathbb{K} \in Mh^\alpha_{N-1}(\varepsilon_{\mathbb{K}+1}) \cap M_{N-1}(Mh^\alpha_{N-1}(\varepsilon_{\mathbb{K}+1}))$.*

Proof.
2.6.1. We show that $A(a)$ is progressive, i.e., $\forall a < \varepsilon_{\mathbb{K}+1}[\forall c < a\, A(c) \to A(a)]$.

Assume $\forall c < a\, A(c)$ and $a < \varepsilon_{\mathbb{K}+1}$. $\forall b < \mathbb{K} \exists ! x[x = \mathcal{H}_a(b)]$ follows from IH in ZFL. $\exists ! b \leq \kappa(b = \psi_\kappa^{\vec{\xi}} a)$ follows from this.

2.6.2. Let π be a weakly inaccessible cardinal with $f\text{"}(\{a\} \cup K(\xi)) \subset L_\pi$. Let f be an injection such that $f\text{"}\mathcal{H}_{\varepsilon_{\mathbb{K}+1}}(\pi) \subset L_\pi$. Then for $\forall \alpha \in K(\xi)(f(\alpha) \in f\text{"}\mathcal{H}_\alpha(\pi))$, $\pi \in Mh_k^a(\xi)$ iff for any $f(\vec{\nu}) = (f(\nu_k), \ldots, f(\nu_{N-1}))$, each of $f(\nu_i) \in L_\pi$, if $\forall \alpha \in K^2(\vec{\nu})(f(\alpha) \in f\text{"}\mathcal{H}_a(\pi))$ and $\vec{\nu} <_{tl} \xi$, then $\pi \in M_k(Mh_k^a(\vec{\nu}))$, where $f\text{"}\mathcal{H}_a(\pi) \subset L_\pi$ is a class in L_π.

2.6.3. We show the following $B(a)$ is progressive in $a < \varepsilon_{\mathbb{K}+1}$:

$$B(a) :\Leftrightarrow \mathbb{K} \in Mh^\alpha_{N-1}(a) \cap M_{N-1}(Mh^\alpha_{N-1}(a))$$

Note that $a \in \mathcal{H}_a(\mathbb{K})$ holds for any $a < \varepsilon_{\mathbb{K}+1}$.

Suppose $\forall b < a\, B(b)$. We have to show that $Mh^\alpha_{N-1}(a)$ is Π^1_{N-3}-indescribable in $\mathbb{K}$. It is easy to see that if $\pi \in M_{N-1}(Mh^\alpha_{N-1}(a))$, then $\pi \in Mh^\alpha_{N-1}(a)$ by induction on π. Let $\theta(u)$ be a Π^1_{N-3}-formula such that $L_\mathbb{K} \models \theta(u)$.

By IH we have $\forall b < a[\mathbb{K} \in M_{N-1}(Mh^\alpha_{N-1}(b))]$. In other words, $\mathbb{K} \in Mh^\alpha_{N-1}(a)$, i.e., $L_\mathbb{K} \models mh^\alpha_{N-1}(a)$, where $mh^\alpha_{N-1}(a)$ is a Π^1_{N-2}-sentence in Proposition 2.6.2. Since the universe $L_\mathbb{K}$ is Π^1_{N-2}-indescribable, pick a $\pi < \mathbb{K}$ such that L_π enjoys the Π^1_{N-2}-sentence $\theta(u) \wedge mh^\alpha_{N-1}(a)$, and $\{f(\alpha), f(a)\} \subset L_\pi$. Therefore $\pi \in Mh^\alpha_{N-1}(a)$ and $L_\pi \models \theta(u)$. Thus $\mathbb{K} \in M_{N-1}(Mh^\alpha_{N-1}(a))$. $\Box$

Proposition 2.7 $\pi \in Mh_k^a(\zeta)\, \&\, \xi \leq \zeta \Rightarrow \pi \in Mh_k^a(\xi)$.

Proof. By the definition (3) of $\pi \in Mh_k^a(\zeta)$, it suffices to show that

$$\vec{\nu} <_{tl} \xi \leq \zeta \Rightarrow \vec{\nu} <_{tl} \zeta$$

by induction on the lengths $n = lh(\vec{\nu})$. Let $\vec{\mu} = (\mu_0, \ldots, \mu_{n-1})$ be a sequence for $\vec{\nu} = (\nu_0, \ldots, \nu_{n-1})$ such that $\mu_0 \leq_{pt} \xi$, $\forall i < n-1(\mu_{i+1} \leq_{pt} te(\mu_i))$, and $\forall i \leq n-1(\nu_i < \mu_i)$, cf. (2) in Definition 2.3.

If $n = 1$, then $\nu_0 < \mu_0 \leq_{pt} \xi \leq \zeta$. $\nu_0 < \zeta \leq_{pt} \zeta$ yields $\vec{\nu} = (\nu_0) <_{tl} \zeta$.

Let $n > 1$. We have $(\nu_1, \ldots, \nu_{n-1}) <_{tl} te(\mu_0)$. We show the existence of a λ such that $\mu_0 \leq \lambda \leq_{pt} \zeta$ and $te(\mu_0) \leq te(\lambda)$. Then IH yields $(\nu_1, \ldots, \nu_{n-1}) <_{tl} te(\lambda)$, and $\vec{\nu} <_{tl} \zeta$ follows.

If $\mu_0 \leq_{pt} \zeta$, then $\lambda = \mu_0$ works. Suppose $\mu_0 \not\leq_{pt} \zeta$. On the other hand we have $\mu_0 \leq_{pt} \xi \leq \zeta$. Hence $\xi < \zeta$ and there exists a $\lambda \leq_{pt} \zeta$ such that $\mu_0 < \lambda$ and $te(\mu_0) \leq te(\lambda)$. □

Lemma 2.8 (Cf. Lemma 3 in [3].)

Assume $\mathbb{K} \geq \pi \in Mh^a_k(\xi) \cap Mh^a_{k+1}(\xi_0)$ *with* $2 \leq k \leq N-1$, $he(\mu) \leq \xi_0$ *and* $\{a\} \cup K(\mu) \subset \mathcal{H}_a(\pi)$. *Then* $\pi \in Mh^a_k(\xi + \mu)$ *holds. Moreover if* $\pi \in M_{k+1}$, *then* $\pi \in M_{k+1}(Mh^a_k(\xi + \mu))$ *holds.*

Proof. Suppose $\pi \in Mh^a_k(\xi) \cap Mh^a_{k+1}(\xi_0)$ and $K(\mu) \subset \mathcal{H}_a(\pi)$ with $he(\mu) \leq \xi_0$. We show $\pi \in Mh^a_k(\xi + \mu)$ by induction on ordinals μ. First note that if $b \in \mathcal{H}_a(\pi)$, then $f(b) \in f\text{"}\mathcal{H}_{\varepsilon_{\mathbb{K}+1}}(\pi) \subset L_\pi$. We have $K(\xi + \mu) \subset \mathcal{H}_a(\pi)$. $\pi \in M_{k+1}(Mh^a_k(\xi + \mu))$ follows from $\pi \in Mh^a_k(\xi + \mu)$ and $\pi \in M_{k+1}$.

Let $(\zeta) * \vec{\nu} <_{tl} \xi + \mu$ and $K(\zeta) \cup K^2(\vec{\nu}) \subset \mathcal{H}_a(\pi)$ for $\vec{\nu} = (\nu_0, \ldots, \nu_{n-1})$. We need to show that $\pi \in M_k(Mh^a_k((\zeta) * \vec{\nu}))$. By the definition (2), let $(\zeta_0) * (\mu_0, \ldots, \mu_{n-1})$ be a sequence such that $\zeta < \zeta_0 \leq_{pt} \xi + \mu$, $\mu_0 \leq_{pt} te(\zeta_0)$, $\forall i \leq n-1(\nu_i < \mu_i)$, and $\forall i < n-1(\mu_{i+1} \leq_{pt} te(\mu_i))$.

If $\zeta_0 \leq_{pt} \xi$, then $(\zeta) * \vec{\nu} <_{tl} \xi$, and $\pi \in M_k(Mh^a_k((\zeta) * \vec{\nu}))$ by $\pi \in Mh^a_k(\xi)$.

Let $\zeta_0 = \xi + \zeta_1$ with $0 < \zeta_1 \leq_{pt} \mu$. If $\zeta_1 <_{pt} \mu$, then by IH with $he(\zeta_1) = he(\mu)$ we have $\pi \in Mh^a_k(\zeta_0)$. On the other hand we have $(\zeta) * \vec{\nu} <_{tl} \zeta_0$. Hence $\pi \in M_k(Mh^a_k((\zeta) * \vec{\nu}))$.

Finally consider the case when $0 < \zeta_1 = \mu$. Then we have $\vec{\nu} <_{tl} te(\xi + \mu) = te(\mu) \leq he(\mu) \leq \xi_0$. $\pi \in Mh^a_{k+1}(\xi_0)$ with Proposition 2.7 yields $\pi \in M_{k+1}(Mh^a_{k+1}(\vec{\nu}))$.

On the other side we see $\pi \in Mh^a_k(\zeta)$ as follows. We have $\zeta < \xi + \mu$. If $\zeta \leq \xi$, then this follows from $\pi \in Mh^a_k(\xi)$ and Proposition 2.7, and if $\zeta = \xi + \lambda < \xi + \mu$, then IH yields $\pi \in Mh^a_k(\zeta)$.

Since $\pi \in Mh^a_k(\zeta)$ is a Π^1_{k-1}-sentence holding on L_π by Lemma 2.6.2 and $\{a\} \cup K(\zeta) \subset \mathcal{H}_a(\pi)$, we obtain $\pi \in M_{k+1}(Mh^a_k((\zeta) * \vec{\nu}))$, a fortiori $\pi \in M_k(Mh^a_k((\zeta) * \vec{\nu}))$. □

Proposition 2.9 *Let* $\vec{\nu} = (\nu_2, \ldots, \nu_{N-1})$, $\vec{\xi} = (\xi_2, \ldots, \xi_{N-1})$ *be sequences of ordinals*$< \varepsilon_{\mathbb{K}+2}$ *such that for a* k *with* $2 \leq k \leq N-1$, $\forall i < k(\nu_i \leq \xi_i)$ *and* $(\nu_k, \ldots, \nu_{N-1}) <_{tl} \xi_k$. *Assume* $\pi \in Mh^a_2(\vec{\xi})$ *and* $K^2(\vec{\nu}) \cup \{\pi, a\} \subset \mathcal{H}_a(\pi)$ *for* $a \geq \max K^2(\vec{\nu})$. *Then* $\psi^{\vec{\nu}}_\pi(a) < \pi$.

Proof. By the definition (4) it suffices to show the existence of a $\kappa \in Mh^a_2(\vec{\nu}) \cap \pi$ such that $\mathcal{H}_a(\kappa) \cap \pi \subset \kappa$ and $K^2(\vec{\nu}) \cup \{\pi, a\} \subset \mathcal{H}_a(\kappa)$. We have $\pi \in Mh^a_k(\xi_k)$ by $\pi \in Mh^a_2(\vec{\xi})$, $K^2(\vec{\nu}) \subset \mathcal{H}_a(\pi)$ and $(\nu_k, \ldots, \nu_{N-1}) <_{tl} \xi_k$. Hence by the definition (3) we obtain $\pi \in M_k(Mh^a_k((\nu_k, \ldots, \nu_{N-1})))$, i.e., $\pi \in M_k(\bigcap_{k \leq i \leq N-1} Mh^a_i(\nu_i))$.

On the other hand we have $\pi \in \bigcap_{i<k} Mh_i^a(\xi_i)$, and hence $\pi \in \bigcap_{i<k} Mh_i^a(\nu_i)$ by $\forall i < k(\nu_i \leq \xi_i)$ and Proposition 2.7. Since $\pi \in \bigcap_{i<k} Mh_i^a(\nu_i)$ is a Π^1_{k-2}-sentence holding in L_π, we obtain $\pi \in M_k(\bigcap_{i\leq N-1} Mh_i^a(\nu_i))$, a fortiori $\pi \in M_2(Mh_2^a(\vec{\nu}))$.

On the other side, since $K^2(\vec{\nu}) \cup \{\pi, a\} \subset \mathcal{H}_a(\pi)$, the set $C = \{\kappa < \pi : \mathcal{H}_a(\kappa) \cap \pi \subset \pi, K^2(\vec{\nu}) \cup \{\pi, a\} \subset \mathcal{H}_a(\kappa)\}$ is a club subset of regular cardinal π. This shows the existence of a $\kappa \in Mh_2^a(\vec{\nu}) \cap C \cap \pi$. $\Box$

Proposition 2.10 *Let $\vec{\xi} = (\xi_2, \ldots, \xi_{N-1})$ be a sequence of ordinals $< \varepsilon_{\mathbb{K}+2}$ such that $K^2(\vec{\xi}) \subset \mathcal{H}_a(\pi)$. If $Tl(\xi_i) < \Lambda_k(\xi_{i+k}+1)$ for some $i < N-1$ and $k > 0$, then*

$$\pi \in Mh_2^a(\vec{\xi}) \Leftrightarrow \pi \in Mh_2^a(\vec{\mu})$$

where $\vec{\mu} = (\mu_2, \ldots, \mu_{N-1})$ with $\mu_i = \xi_i - Tl(\xi_i)$ and $\mu_j = \xi_j$ for $j \neq i$.

Proof. When $0 < \xi_i = \Lambda^{\gamma_m} a_m + \cdots + \Lambda^{\gamma_1} a_1 + \Lambda^{\gamma_0} a_0$ with $\gamma_m > \cdots > \gamma_1 > \gamma_0$, $0 < a_i < \Lambda$, $\mu_i = \Lambda^{\gamma_m} a_m + \cdots + \Lambda^{\gamma_1} a_1$ for $Tl(\xi_i) = \Lambda^{\gamma_0} a_0$. If $\xi_i = 0$, then so is $\mu_i = 0$.

Let $\pi \in Mh_2^a(\vec{\mu})$ and $Tl(\xi_i) < \Lambda_k(\xi_{i+k}+1)$. We have $\forall j \leq k(he^{(j)}(Tl(\xi_i)) < \Lambda_{k-j}(\xi_{i+k}+1))$, and $he^{(k)}(Tl(\xi_i)) \leq \xi_{i+k}$. On the other hand we have $\pi \in Mh_{i+k}^a(\xi_{i+k})$. From Lemma 2.8 we see inductively that for any $j < k$, $\pi \in Mh_{i+j}^a(he^{(j)}(Tl(\xi_i)))$. In particular $\pi \in Mh_{i+1}^a(he(Tl(\xi_i)))$, and once again by Lemma 2.8 and $\pi \in Mh_i^a(\mu_i)$ we obtain $\pi \in Mh_i^a(\xi_i)$. Hence $\pi \in Mh_2^a(\vec{\xi})$. $\Box$

Definition 2.11 *1.* A sequence of ordinals $\vec{\xi} = (\xi_2, \ldots, \xi_{N-1})$ is said to be *irreducible* iff $\forall i < N-1 \forall k > 0(\xi_i > 0 \Rightarrow Tl(\xi_i) \geq \Lambda_k(\xi_{i+k}+1))$.

2. For sequences of ordinals $\vec{\xi} = (\xi_k, \ldots, \xi_{N-1})$ and $\vec{\nu} = (\nu_k, \ldots, \nu_{N-1})$ and $2 \leq k \leq N-1$,

$$Mh_k^a(\vec{\nu}) \prec_k Mh_k^a(\vec{\xi}) :\Leftrightarrow \forall \pi \in Mh_k^a(\vec{\xi})(K^2(\vec{\nu}) \subset \mathcal{H}_a(\pi) \Rightarrow \pi \in M_k(Mh_k^a(\vec{\nu}))).$$

Definition 2.12 Let $\vec{\xi} = (\xi_k, \ldots, \xi_{N-1})$, $\vec{\nu} = (\nu_k, \ldots, \nu_{N-1})$ and $\vec{\nu} \neq \vec{\xi}$. Let $i \geq k$ be the minimal number such that $\nu_i \neq \xi_i$. Suppose $(\xi_i, \ldots, \xi_{N-1}) \neq \vec{0}$, and let $k_1 \geq i$ be the minimal number such that $\xi_{k_1} \neq 0$. Then $\vec{\nu} <_{lx,k} \vec{\xi}$ iff one of the followings holds:

1. $(\nu_i, \ldots, \nu_{N-1}) = \vec{0}$.

2. In what follows assume $(\nu_i, \ldots, \nu_{N-1}) \neq \vec{0}$, and let $k_0 \geq i$ be the minimal number such that $\nu_{k_0} \neq 0$ $(i = \min\{k_0, k_1\})$. Then $\vec{\nu} <_{lx,k} \vec{\xi}$ iff one of the followings holds:

   (a) $i = k_0 < k_1$ and $he^{(k_1-i)}(\nu_i) \leq \xi_{k_1}$.

   (b) $k_0 \geq k_1 = i$ and $\nu_{k_0} < he^{(k_0-i)}(\xi_i)$.

We write $<_{lx}$ for $<_{lx,2}$.

Proposition 2.13 *Suppose that both of $\vec{\nu}$ and $\vec{\xi}$ are irreducible. Then*

$$\vec{\nu} <_{lx,k} \vec{\xi} \Rightarrow Mh^a_k(\vec{\nu}) \prec_k Mh^a_k(\vec{\xi}).$$

Proof.

Let $\pi \in Mh^a_k(\vec{\xi})$, $K^2(\vec{\nu}) \subset \mathcal{H}_a(\pi)$, and $i \geq k$ be the minimal number such that $\nu_i \neq \xi_i$. First we have $\pi \in \bigcap_{k\leq j<i} Mh^a_j(\nu_j)$, which is a Π^1_{i-2}-sentence holding on L_π. In the case $\xi_i \neq 0$, it suffices to show that $\pi \in M_i(\bigcap_{j\geq i} Mh^a_j(\nu_j))$, since then we obtain $\pi \in M_i(Mh^a_k(\vec{\nu}))$ by $\pi \in Mh^a_i(\xi_i) \subset M_i$, a fortiori $\pi \in M_k(Mh^a_k(\vec{\nu}))$.

If $(\nu_i, \ldots, \nu_{N-1}) = \vec{0}$, then $\xi_i \neq 0$ and $\bigcap_{j\geq i} Mh^a_j(\nu_j)$ denotes the class of limit ordinals. Obviously $\pi \in M_i(\bigcap_{j\geq i} Mh^a_j(\nu_j))$.

In what follows assume $(\nu_i, \ldots, \nu_{N-1}) \neq \vec{0}$, and let $k_0 \geq i$ be the minimal number such that $\nu_{k_0} \neq 0$, and $k_1 \geq i$ be the minimal number such that $\xi_{k_1} \neq 0$.

First consider the case when $i = k_0 = k_1$. Then $0 < \nu_i < \xi_i$. It suffices to show that $(\nu_i, \ldots, \nu_{N-1}) <_{tl} \xi_i$, since then we have $\pi \in M_i(\bigcap_{j\geq i} Mh^a_j(\nu_j))$ by the definition (3) of $\pi \in Mh^a_i(\xi_i)$. Suppose that $(\nu_{i+1}, \ldots, \nu_{N-1}) \neq \vec{0}$, and let $0 < \ell \leq N-1-i$ be the least number such that $\nu_{i+\ell} > 0$. Since $\vec{\nu}$ is irreducible, we have $Tl(\nu_i) \geq \Lambda_\ell(\nu_{i+\ell}+1)$, and hence $he^{(\ell)}(Tl(\nu_i)) > \nu_{i+\ell}$. On the other hand we have $he^{(\ell)}(Tl(\nu_i)) \leq he^{(\ell)}(\xi_i)$.

Therefore for $\mu_i = Hd(\xi_i)$, $\mu_{j+1} = Hd(he(\mu_j))$ for $j < \ell - i$, we have $\mu_i \leq_{pt} \xi_i$, $\forall j < \ell - i(\mu_{j+1} \leq_{pt} he(\mu_j) = te(\mu_j))$ and $\nu_{i+\ell} < he^{(\ell)}(\xi_i) = \mu_\ell$. In this way we see the existence of a sequence $(\mu_i, \ldots, \mu_{N-1})$ witnessing $(\nu_i, \ldots, \nu_{N-1}) <_{tl} \xi_i$ in the definition (2).

Second assume $i = k_0 < k_1$. Then we have $he^{(k_1-i)}(\nu_i) \leq \xi_{k_1}$. Also $\nu_{i+p} < he^{(p)}(\nu_i)$ for any $p > 0$ since $\vec{\nu}$ is irreducible and $\nu_i \neq 0$. Let $j \geq k_1$. Then $\nu_j < he^{(j-i)}(\nu_i) \leq he^{(j-k_1)}(\xi_{k_1})$. Hence $(\nu_{k_1}, \ldots, \nu_{N-3}) <_{tl} \xi_{k_1}$ by the definition (2). $\pi \in Mh^a_{k_1}(\xi_{k_1})$ yields $\pi \in M_{k_1}(\bigcap_{j\geq k_1} Mh^a_j(\nu_j))$. Moreover for any $p < k_1 - i$, $he^{(k_1-i-p)}(\nu_{i+p}) \leq \xi_{k_1}$ by Proposition 2.2. Lemma 2.8 yields $\pi \in \bigcap_{k_1>j\geq i} Mh^a_j(\nu_j)$. Therefore $\pi \in M_{k_1}(Mh^a_k(\vec{\nu}))$, a fortiori $\pi \in M_k(Mh^a_k(\vec{\nu}))$.

Finally assume $k_0 > k_1 = i$. Then we have $\xi_i \neq 0$. It suffices to show that $(\nu_i, \ldots, \nu_{N-1}) <_{tl} \xi_i$. We have $\nu_{k_0} < he^{(k_0-i)}(\xi_i)$ and $\forall k > k_0(\nu_k < he^{(k-k_0)}(\nu_{k_0}) \leq he^{(k-i)}(\xi_i))$, where $he^{(k)}(\xi_i) = te(Hd(hd^{(k-1)}(\xi_i)))$. Therefore $(\nu_i, \ldots, \nu_{N-1}) <_{tl} \xi_i$ by the definition (2). □

Definition 2.14 Let us define an ordinal $o(\vec{\xi})$ for irreducible $\vec{\xi} = (\xi_2, \ldots, \xi_{N-1})$ by

$$o(\vec{\xi}) = \sum\{\Lambda_{i-1}(\xi_i+1) : 2 \leq i \leq N-1, \xi_i \neq 0\}$$

In particular $o(\vec{0}) = 0$.

Note that we have $Tl(\xi_i) \geq \Lambda_k(\xi_{i+k}+1)$ for $\xi_i \neq 0$ and irreducible $\vec{\xi}$. Therefore $\xi_i + \Lambda_k(\xi_{i+k}+1) = \xi_i \# \Lambda_k(\xi_{i+k}+1)$ for the natural sum $\#$.

Proposition 2.15 *For irreducible $\vec{\nu}, \vec{\xi}$,*

$$\vec{\nu} <_{lx} \vec{\xi} \Rightarrow o(\vec{\nu}) < o(\vec{\xi}).$$

Proof. Let $\vec{\nu} <_{lx} \vec{\xi}$. Then $\vec{\nu} \neq \vec{\xi}$ and let i be the minimal number such that $\nu_i \neq \xi_i$. It suffices to show that $a_0 = o((\nu_i, \ldots, \nu_{N-1})) < o((\xi_i, \ldots, \xi_{N-1})) = a_1$, where $o((\xi_i, \ldots, \xi_{N-1})) = \sum\{\Lambda_{j-1}(\xi_j + 1) : i \leq j \leq N-1, \xi_j \neq 0\}$.

We have $(\xi_i, \ldots, \xi_{N-1}) \neq \vec{0}$, and let $k_1 \geq i$ be the minimal number such that $\xi_{k_1} \neq 0$. When $(\nu_i, \ldots, \nu_{N-1}) \neq \vec{0}$, let $k_0 \geq i$ be the minimal number such that $\nu_{k_0} \neq 0$. One of the following cases occurs, cf. Definition 2.12.

Case 0. $(\nu_i, \ldots, \nu_{N-1}) = \vec{0}$: Then $a_0 = 0 < a_1$.

Case 1. $i = k_0 < k_1 = i + k$ and $he^{(k)}(\nu_i) \leq \xi_{i+k}$: We have by $k > 0$
$o((\nu_i, \ldots, \nu_{N-1})) = \Lambda_{i-1}(\nu_i + 1) + o((\nu_{i+1}, \ldots, \nu_{N-1})) < \Lambda_{i+k-1}(he^{(k)}(\nu_i) + 1)$.
On the other hand we have $o((\xi_i, \ldots, \xi_{N-1})) = o((0, \ldots, 0, \xi_{i+k}, \ldots, \xi_{N-1})) = \Lambda_{i+k-1}(\xi_{i+k} + 1) + o(\xi_{i+k+1}, \ldots, \xi_{N-1}) \geq \Lambda_{i+k-1}(\xi_{i+k} + 1)$. Hence $a_0 < a_1$.

Case 2. $i + k = k_0 \geq k_1 = i$ and $\nu_{i+k} < he^{(k)}(\xi_i)$: Then

$$\begin{aligned} o((\nu_i, \ldots, \nu_{N-1})) &= o((0, \ldots, 0, \nu_{i+k}, \ldots, \nu_{N-1})) \\ &= \Lambda_{i+k-1}(\nu_{i+k} + 1) + o((\nu_{i+k+1}, \ldots, \nu_{N-1})) \\ &< \Lambda_{i+k-1}(\nu_{i+k} + 1) \cdot 2 \leq \Lambda_{i+k-1}(he^{(k)}(\xi_i)) \cdot 2 \end{aligned}$$

On the other hand we have by $i > 1$ and $\xi_i \geq \Lambda_k(he^{(k)}(\xi_i))$

$$\begin{aligned} o((\xi_i, \ldots, \xi_{N-1})) &= \Lambda_{i-1}(\xi_i + 1) + o((\xi_{i+1}, \ldots, \xi_{N-1})) \\ &\geq \Lambda_{i-1}(\xi_i + 1) > \Lambda_{i+k-1}(he^{(k)}(\xi_i)) \cdot 2 \end{aligned}$$

Hence $a_0 < a_1$. □

Proposition 2.16 *Suppose $\zeta < \mu$ and $\xi \leq te(\mu)$. Then $\zeta + \Lambda^{\xi} \leq \mu$.*

Proof. We have $\Lambda^{\xi} \leq Tl(\mu) = \Lambda^{te(\mu)}a$ for an $a > 0$. If $\zeta \leq \mu_0$ with $\mu = \mu_0 + Tl(\mu)$, then $\zeta + \Lambda^{\xi} \leq \mu$. Otherwise $\zeta = \mu_0 + \Lambda^{te(\zeta)}b$ for $te(\zeta) \leq te(\mu)$, and $b < a$ if $te(\zeta) = te(\mu)$. Hence $\Lambda^{tl(\zeta)}b + \Lambda^{\xi} \leq \Lambda^{te(\mu)}a$. □

Proposition 2.17 (Cf. Proposition 4.20 in [13])
Let $\vec{\nu} = (\nu_2, \ldots, \nu_{N-1})$, $\vec{\xi} = (\xi_2, \ldots, \xi_{N-1})$ be irreducible sequences of ordinals$<$ $\varepsilon_{\mathbb{K}+2}$, and assume that $\psi_{\pi}^{\vec{\nu}}(b) < \pi$ and $\psi_{\kappa}^{\vec{\xi}}(a) < \kappa$.

Then $\beta_1 = \psi_{\pi}^{\vec{\nu}}(b) < \psi_{\kappa}^{\vec{\xi}}(a) = \alpha_1$ iff one of the following cases holds:

1. *$\pi \leq \psi_{\kappa}^{\vec{\xi}}(a)$.*
2. *$b < a$, $\psi_{\pi}^{\vec{\nu}}(b) < \kappa$ and $K^2(\vec{\nu}) \cup \{\pi, b\} \subset \mathcal{H}_a(\psi_{\kappa}^{\vec{\xi}}(a))$.*
3. *$b > a$ and $K^2(\vec{\xi}) \cup \{\kappa, a\} \not\subset \mathcal{H}_b(\psi_{\pi}^{\vec{\nu}}(b))$.*
4. *$b = a$, $\kappa < \pi$ and $\kappa \notin \mathcal{H}_b(\psi_{\pi}^{\vec{\nu}}(b))$.*
5. *$b = a$, $\pi = \kappa$, $K^2(\vec{\nu}) \subset \mathcal{H}_a(\psi_{\kappa}^{\vec{\xi}}(a))$, and $\vec{\nu} <_{lx} \vec{\xi}$.*
6. *$b = a$, $\pi = \kappa$, $K^2(\vec{\xi}) \not\subset \mathcal{H}_b(\psi_{\pi}^{\vec{\nu}}(b))$.*

Proof. If the case (2) holds, then $\psi_{\pi}^{\vec{\nu}}(b) \in \mathcal{H}_a(\psi_{\kappa}^{\vec{\xi}}(a)) \cap \kappa \subset \psi_{\kappa}^{\vec{\xi}}(a)$.

If one of the cases (3) and (4) holds, then $K^2(\vec{\xi}) \cup \{\kappa, a\} \not\subset \mathcal{H}_a(\psi_{\pi}^{\vec{\nu}}(b))$. On the other hand we have $K^2(\vec{\xi}) \cup \{\kappa, a\} \subset \mathcal{H}_a(\psi_{\kappa}^{\vec{\xi}}(a))$. Hence $\psi_{\pi}^{\vec{\nu}}(b) < \psi_{\kappa}^{\vec{\xi}}(a)$.

If the case (5) holds, then by Proposition 2.13 yields $Mh_2^a(\vec{\nu}) \prec_2 Mh_2^a(\vec{\xi}) \ni \psi_{\kappa}^{\vec{\xi}}(a)$. Hence $\psi_{\kappa}^{\vec{\xi}}(a) \in M_2(Mh_2^a(\vec{\nu}))$. Since $K^2(\vec{\nu}) \cup \{\kappa, a\} \subset \mathcal{H}_a(\psi_{\kappa}^{\vec{\xi}}(a))$, the set $\{\rho < \psi_{\kappa}^{\vec{\xi}}(a) : \mathcal{H}_a(\rho) \cap \kappa \subset \rho, K^2(\vec{\nu}) \cup \{\kappa, a\} \subset \mathcal{H}_a(\rho)\}$ is club in $\psi_{\kappa}^{\vec{\xi}}(a)$. Therefore $\psi_{\pi}^{\vec{\nu}}(b) = \psi_{\kappa}^{\vec{\nu}}(a) < \psi_{\kappa}^{\vec{\xi}}(a)$ by the definition (4).

Finally assume that the case (6) holds. Since $K^2(\vec{\xi}) \subset \mathcal{H}_a(\psi_{\kappa}^{\vec{\xi}}(a))$, $\psi_{\pi}^{\vec{\nu}}(b) < \psi_{\kappa}^{\vec{\xi}}(a)$.

Conversely assume that $\psi_{\pi}^{\vec{\nu}}(b) < \psi_{\kappa}^{\vec{\xi}}(a)$ and $\psi_{\kappa}^{\vec{\xi}}(a) < \pi$.

First consider the case $b < a$. Then we have $K^2(\vec{\nu}) \cup \{\pi, b\} \subset \mathcal{H}_b(\psi_{\pi}^{\vec{\nu}}(b)) \subset \mathcal{H}_a(\psi_{\kappa}^{\vec{\xi}}(a))$. Hence (2) holds.

Next consider the case $b > a$. $K^2(\vec{\xi}) \cup \{\kappa, a\} \subset \mathcal{H}_b(\psi_{\pi}^{\vec{\nu}}(b))$ would yield $\psi_{\kappa}^{\vec{\xi}}(a) \in \mathcal{H}_b(\psi_{\pi}^{\vec{\nu}}(b)) \cap \pi \subset \psi_{\pi}^{\vec{\nu}}(b)$, a contradiction $\psi_{\kappa}^{\vec{\xi}}(a) < \psi_{\pi}^{\vec{\nu}}(b)$. Hence (3) holds.

Finally assume $b = a$. Consider the case $\kappa < \pi$. $\kappa \in \mathcal{H}_b(\psi_{\pi}^{\vec{\nu}}(b)) \cap \pi$ would yield $\psi_{\kappa}^{\vec{\xi}}(a) < \kappa < \psi_{\pi}^{\vec{\nu}}(b)$, a contradiction. Hence $\kappa \not\in \mathcal{H}_b(\psi_{\pi}^{\vec{\nu}}(b))$, and (4) holds. If $\pi < \kappa$, then $\pi \in \mathcal{H}_b(\psi_{\pi}^{\vec{\nu}}(b)) \cap \kappa \subset \mathcal{H}_a(\psi_{\kappa}^{\vec{\xi}}(a)) \cap \kappa$, and $\pi < \psi_{\kappa}^{\vec{\xi}}(a)$, a contradiction, or we should say that (1) holds. Finally let $\pi = \kappa$. We can assume that $K^2(\vec{\xi}) \subset \mathcal{H}_b(\psi_{\pi}^{\vec{\nu}}(b))$, otherwise (6) holds. If $\vec{\xi} <_{lx} \vec{\nu}$, then by (5) $\psi_{\kappa}^{\vec{\xi}}(a) < \psi_{\pi}^{\vec{\nu}}(b)$ would follow. If $K^2(\vec{\nu}) \not\subset \mathcal{H}_a(\psi_{\kappa}^{\vec{\xi}}(a))$, then by (6) again $\psi_{\kappa}^{\vec{\xi}}(a) < \psi_{\pi}^{\vec{\nu}}(b)$ would follow. Hence $K^2(\vec{\nu}) \subset \mathcal{H}_a(\psi_{\kappa}^{\vec{\xi}}(a))$ and $\vec{\nu} \leq_{lx} \vec{\xi}$. If $\vec{\nu} = \vec{\xi}$, then $\psi_{\kappa}^{\vec{\xi}}(a) = \psi_{\pi}^{\vec{\nu}}(b)$. Therefore (5) must be the case. □

3 Computable notation system OT

In this section (except Propositions 3.5) we work in a weak fragment of arithmetic, e.g., in the fragment $I\Sigma_1$ or even in the bounded arithmetic S_2^1. Referring Proposition 2.17 the sets of ordinal terms $OT \subset \Lambda = \varepsilon_{\mathbb{K}+1}$ and $E \subset \varepsilon_{\mathbb{K}+2}$ over symbols $\{0, \mathbb{K}, \Lambda, +, \omega, \varphi, \Omega, \psi\}$ together with sequences $(m_k(\alpha))_{2 \leq k \leq N-1}$ for $\alpha \in OT \cap \mathbb{K}$, and finite sets $K_\delta(\alpha) \subset OT$ for $\alpha \in OT$ are defined by simultaneous recursion.

OT is isomorphic to a subset of $\mathcal{H}_\Lambda(0)$, which is restricted with respect to iterated collapsings, i.e., introducing ordinals of the form $\psi_{\pi}^{\vec{\nu}}(a)\,(\vec{\nu} \neq \vec{0}, lh(\vec{\nu}) = N-2)$ as follows. First $\mathbb{K}$ is collapsed only to ordinals $\psi_{\mathbb{K}}^{\vec{0}*(b)}(a)$ with a single ordinal $b < \Lambda$, cf. Definition 3.3.14 below. Second each ordinal $\pi = \psi_{\kappa}^{\vec{\xi}*(\xi_k, \xi_{k+1})*\vec{0}}(c)$ is collapsed only to ordinals $\psi_{\pi}^{\vec{\xi}*(\xi_k+\Lambda^{\xi_{k+1}} b, 0)*\vec{0}}(a)$ with an ordinal $b < \Lambda$. Actually the ordinal b is decorated with the ordinals π, a, denoted as a triple $\langle b, \pi, a\rangle$ to keep track of the point (π, a) at which a new collapsing process $b > b' > \cdots$ in the k-th level starts, cf. Definition 3.3.15. Third each ordinal $\pi = \psi_{\kappa}^{(\xi)*\vec{0}}(c)$ is collapsed only to ordinals $\psi_{\pi}^{\vec{\nu}}(a)$, where $\vec{\nu} <_{tl} \xi$ holds in an explicit way:

let $\vec{\mu} = (\mu_2, \ldots, \mu_{N-1})$ be a witness for $\vec{\nu} <_{tl} \xi$. This means that, cf. (2), $\nu_2 < \mu_2 \leq_{pt} \xi$, $\nu_3 < \mu_3 \leq_{pt} te(\mu_2)$, etc. Here ν_i has to be smaller than μ_i in such a way that ν_i's last coefficient is smaller than that of μ_i's in a strong sense. Namely $\nu_i = \lambda + \Lambda^\alpha x$ and $\mu_i = \lambda + \Lambda^\alpha y$ with $x < y$. Moreover the ordinal x is constructed at the same time when y is introduced. For example let us assume that an ordinal $\psi_\rho^{\cdots*(\mu_i)*\vec{0}}(d)$ is constructed before π in the iterated collapsing process, where y is decorated with (ρ, d). Then $y \in \mathcal{H}_d(\rho)$ holds. The coincidence of y and x means that $x \in \mathcal{H}_d(\pi)$ should be the case, and $\psi_{\rho^+}(x) < \psi_{\rho^+}(y)$, cf. Definition 3.3.16.

Non-zero terms in E denotes ordinals$< \varepsilon_{\mathbb{K}+2}$ in Cantor normal form with base Λ, which are decorated by indicators (π_i, a_i). The triple $\langle \alpha, \mathbb{K}, a \rangle$ in Definition 3.3.3 denotes the ordinal α, and in Definition 3.3.4, $\Lambda^{\xi_m} \langle b_m, \pi_m, a_m \rangle + \cdots + \Lambda^{\xi_0} \langle b_0, \pi_0, a_0 \rangle$ denotes the ordinal $\Lambda^{\xi_m} b_m + \cdots + \Lambda^{\xi_0} b_0$.

$\alpha =_{NF} \alpha_m + \cdots + \alpha_0$ means that $\alpha = \alpha_m + \cdots + \alpha_0$ and $\alpha_m \geq \cdots \geq \alpha_0$ and each α_i is a non-zero additive principal number. $\alpha =_{NF} \varphi\beta\gamma$ means that $\alpha = \varphi\beta\gamma$ and $\beta, \gamma < \alpha$. $\alpha =_{NF} \omega^\beta$ means that $\alpha = \omega^\beta > \beta$. $\alpha =_{NF} \Omega_\beta$ means that $\alpha = \Omega_\beta > \beta$.

Definition 3.1 *For the ordinal* $\xi =_{NF} \Lambda^{\xi_m} \langle b_m, \pi_m, a_m \rangle + \cdots + \Lambda^{\xi_1} \langle b_1, \pi_1, a_1 \rangle + \Lambda^{\xi_0} \langle b_0, \pi_0, a_0 \rangle$,

1. *Let* $st(\xi) := \langle b_0, \pi_0, a_0 \rangle$. *Also* $st(0) := 0$, *and* $te(\xi) = \xi_0$.

2. (a) $hd(\xi) = \sum_{i \geq 1} \Lambda^{\xi_i} \langle b_i, \pi_i, a_i \rangle = \Lambda^{\xi_m} \langle b_m, \pi_m, a_m \rangle + \cdots + \Lambda^{\xi_1} \langle b_1, \pi_1, a_1 \rangle$ *is the* head part *of* ξ. $\xi = hd(\xi) + Tl(\xi)$ *with* $Tl(\xi) = \Lambda^{\xi_0} \langle b_0, \pi_0, a_0 \rangle$. *Also* $hd(0) := 0$.

   (b) $hd^{(n)}(\xi) := \sum_{i \geq n} \Lambda^{\xi_i} \langle b_i, \pi_i, a_i \rangle = \Lambda^{\xi_m} \langle b_m, \pi_m . a_m \rangle + \cdots + \Lambda^{\xi_n} \langle b_n, \pi_n, a_n \rangle$. *When* $n > m$, $hd^{(n)}(\xi) = \sum_{i \geq n} \Lambda^{\xi_i} \langle b_i, \pi_i, a_i \rangle := 0$.

   (c) *Also for sequences* $\vec{n}$ *of natural numbers* $hd^{(\vec{n})}(\xi)$ *is defined recursively by* $hd^{(\emptyset)}(\xi) := \xi$, $hd^{((n))}(\xi) := hd^{(n)}(\xi)$, *and for* $\vec{n} \neq \emptyset$ $hd^{((n)*\vec{n})}(\xi) := hd^{(\vec{n})}(te(hd^{(n)}(\xi)))$.

3. $\zeta \leq_{pt} \xi :\Leftrightarrow \exists n(\zeta = hd^{(n)}(\xi))$. $\zeta <_{pt} \xi :\Leftrightarrow \zeta \leq_{pt} \xi \,\&\, \zeta \neq \xi$.

4. $\nu <_{st} \xi$ *iff either* $\nu = 0 \neq \xi$ *or for* $c = st(\nu) < st(\xi) = b_0$, $\nu =_{NF} \Lambda^{\xi_m} \langle b_m, \pi_m, a_m \rangle + \cdots + \Lambda^{\xi_1} \langle b_1, \pi_1, a_1 \rangle + \Lambda^{\xi_0} \langle c, \pi_0, a_0 \rangle$.

5. $\nu <_0 \xi$ *iff there is an* $n \leq m+1$ *such that* $\nu =_{NF} \Lambda^{\xi_m} \langle b_m, \pi_m, a_m \rangle + \cdots + \Lambda^{\xi_{n+1}} \langle b_{n+1}, \pi_{n+1}, a_{n+1} \rangle + \Lambda^{\xi_n} \langle c_n, \pi_n, a_n \rangle + \Lambda^{\nu_{n-1}} \langle c_{n-1}, \rho_{n-1}, d_{n-1} \rangle + \cdots$ *with* $c_n < b_n$.

6. (Cf. Definition 2.11.) *A sequence of ordinals* $\vec{\xi} = (\xi_2, \ldots, \xi_{N-1})$ *is said to be* strongly irreducible *iff* $\forall i < N-1 \forall j > i \forall k(\xi_i \neq 0 \wedge \xi_j \neq 0 \wedge i < k < j \Rightarrow \xi_j <_0 te^{(j-i)}(\xi_i) \wedge \xi_k \neq 0)$.

Obviously any strongly irreducible sequence is irreducible.

Proposition 3.2 (Cf. Proposition 2.16.)
For a strongly irreducible sequence $\vec{\xi} = (\xi_2, \ldots, \xi_k, \xi_{k+1}) * \vec{0}$ with $2 \leq k < N-1$ and $\xi_{k+1} \neq 0$, let $\vec{\zeta} = (\zeta_2, \ldots, \zeta_k) * \vec{0}$ be the sequence defined by $\forall i < k(\zeta_i = \xi_i)$ and $\zeta_k = \xi_k + \Lambda^{\xi_{k+1}}\langle b, \pi, a\rangle$ for ordinals $b, a < \Lambda$ and $\pi < \mathbb{K}$. Then $\vec{\zeta}$ is strongly irreducible.

Proof. Supposing $\xi_i \neq 0$ for $i < k$, it suffices to show that $\zeta_k <_0 te^{(k-i)}(\xi_i)$. We have $\xi_k <_0 te^{(k-i)}(\xi_i)$ and $\xi_k \neq 0$. Hence $\zeta_k = \xi_k + \Lambda^{\xi_{k+1}}\langle b, \pi, a\rangle <_0 te^{(k-i)}(\xi_i)$. □

Let $pd(\psi_\pi^{\vec{\nu}}(a)) = \pi$ (even if $\vec{\nu} = \vec{0}$). Moreover for n, $pd^{(n)}(\alpha)$ is defined recursively by $pd^{(0)}(\alpha) = \alpha$ and $pd^{(n+1)}(\alpha) \simeq pd(pd^{(n)}(\alpha))$.

For terms $\pi, \kappa \in OT$, $\pi \prec \kappa$ denotes the transitive closure of the relation $\{(\pi, \kappa) : \exists\vec{\xi}\exists b[\pi = \psi_\kappa^{\vec{\xi}}(b)]\}$, and its reflexive closure $\pi \preceq \kappa :\Leftrightarrow \pi \prec \kappa \vee \pi = \kappa$.

Definition 3.3 *1.* $\ell\alpha$ denotes the number of occurrences of symbols $\{0, \mathbb{K}, \Lambda, +, \omega, \varphi, \Omega, \psi\}$ in terms $\alpha \in OT \cup E$.

2. $0 \in E$.

3. If $0 < \alpha \in OT$ and $\alpha \leq a \in OT$, then $\langle \alpha, \mathbb{K}, a\rangle \in E$. $K(\langle \alpha, \mathbb{K}, a\rangle) = \{\langle \alpha, \mathbb{K}, a\rangle\}$.

4. If $\{\xi_i : i \leq m\} \subset E$, $\xi_m > \cdots > \xi_0 > 0$ and $b_i, \pi_i, a_i \in OT$ with $a_i \geq b_i > 0$, then $\sum_{i \leq m} \Lambda^{\xi_i}\langle b_i, \pi_i, a_i\rangle = \Lambda^{\xi_m}\langle b_m, \pi_m, a_m\rangle + \cdots + \Lambda^{\xi_0}\langle b_0, \pi_0, a_0\rangle \in E$. $K(\sum_{i \leq m} \Lambda^{\xi_i}\langle b_i, \pi_i, a_i\rangle) = \{\langle b_i, \pi_i, a_i\rangle : i \leq m\} \cup \bigcup\{K(\xi_i) : i \leq m\}$.

5. For sequences $\vec{\nu} = (\nu_0, \ldots, \nu_n)$ of $\nu_i \in E$, $K^2(\vec{\nu}) = \bigcup\{K(\nu_i) : i \leq n\}$.

6. (Cf. Definition 3.1.4.)

For $\nu, \mu \in E$, $\nu <_{Kst} \mu$ iff one of the following conditions is fulfilled:

(a) $\nu = 0 \neq \mu$.

(b) $\nu = \langle \alpha, \mathbb{K}, a\rangle, \mu = \langle \beta, \mathbb{K}, b\rangle$ with $\alpha < \beta$.

(c) $\nu =_{NF} \Lambda^{\xi_m}\langle b_m, \pi_m, a_m\rangle + \cdots + \Lambda^{\xi_1}\langle b_1, \pi_1, a_1\rangle + \Lambda^{\xi_0}\langle b, \pi_0, a_0\rangle$ and $\mu =_{NF} \Lambda^{\xi_m}\langle b_m, \pi_m, a_m\rangle + \cdots + \Lambda^{\xi_1}\langle b_1, \pi_1, a_1\rangle + \Lambda^{\xi_0}\langle c, \pi_0, a_0\rangle$ with $b = st(\nu) < st(\mu) = c$ and for $\alpha = \psi_{\pi_0^+} b, \beta = \psi_{\pi_0^+} c$

$$K_\alpha(\{\pi_0, b\}) < b \,\&\, K_\beta(\{\pi_0, c\}) < c \,\&\, \psi_{\pi_0^+} b < \psi_{\pi_0^+} c \tag{5}$$

where $\psi_{\pi_0^+} b := b$ when $\pi_0 = \mathbb{K}$.

$\nu \leq_{Kst} \mu :\Leftrightarrow \nu <_{Kst} \mu \vee \nu = \mu$.

7. (Cf. (2) in Definition 2.3.2.)

Let for sequences $\vec{\nu} = (\nu_2, \ldots, \nu_{n-1})\,(n > 2)$ of terms in E and $\xi \in E$, $\vec{\nu} <_{Ksl} \xi$ iff there exists a sequence $\vec{p} = (p_k)_{2 \leq k \leq n-1}$ of natural numbers such that

$$\forall i \leq n-1[\nu_i <_{Kst} hd^{(\vec{p}_i)}(\xi)]$$

where $\vec{p}_i = (p_k)_{2 \leq k \leq i}$.

8. $0, \mathbb{K} \in OT$. $m_k(0) = 0$, and $K_\delta(0) = K_\delta(\mathbb{K}) = \emptyset$.

9. If $\alpha =_{NF} \alpha_m + \cdots + \alpha_0 \, (m > 0)$ with $\{\alpha_i : i \leq m\} \subset OT$, then $\alpha \in OT$, and $m_k(\alpha) = 0$. $K_\delta(\alpha) = \bigcup_{i \leq m} K_\delta(\alpha_i)$.

10. If $\alpha =_{NF} \varphi\beta\gamma$ with $\{\beta, \gamma\} \subset OT \cap \mathbb{K}$, then $\alpha \in OT$, and $m_k(\alpha) = 0$. $K_\delta(\alpha) = K_\delta(\beta) \cup K_\delta(\gamma)$.

11. If $\alpha =_{NF} \omega^\beta$ with $\mathbb{K} < \beta \in T$, then $\alpha \in OT$, and $m_k(\alpha) = 0$. $K_\delta(\alpha) = K_\delta(\beta)$.

12. If $\alpha =_{NF} \Omega_\beta$ with $\beta \in OT \cap \mathbb{K}$, then $\alpha \in OT$. $m_2(\alpha) = 1, m_k(\alpha) = 0$ for any $k > 2$ if β is a successor ordinal. Otherwise $m_k(\alpha) = 0$ for any k. In each case $K_\delta(\alpha) = K_\delta(\beta)$.

13. Let $\alpha = \psi_\pi(a) := \psi_\pi^{\vec{0}}(a)$ where either $\pi = \mathbb{K}$ or $(m_i(\pi))_i \neq \vec{0}$, and such that $K_\alpha(\pi) \cup K_\alpha(a) < a$, then $\alpha = \psi_\pi(a) \in OT$ and $m_k(\alpha) = 0$ for any $k \geq 2$.

    $K_\delta(\psi_\pi(a)) = \emptyset$ if $\alpha < \delta$. $K_\delta(\psi_\pi(a)) = \{a\} \cup K_\delta(a) \cup K_\delta(\pi)$ otherwise.

14. Let $\alpha = \psi_{\mathbb{K}}^{\vec{\nu}}(a)$ with $\vec{\nu} = \vec{0} * (\langle b, \mathbb{K}, a \rangle) \, (lh(\vec{\nu}) = N - 2)$, $b, a \in OT$ and $\nu \in OT$ such that $K_\alpha(b) < b \leq a$ and $K_\alpha(a) < a$. Then $\alpha = \psi_{\mathbb{K}}^{\vec{\nu}}(a) \in OT$, and $m_{N-1}(\alpha) = \langle b, \mathbb{K}, a \rangle$, $m_k(\alpha) = 0$ for $k < N - 1$.

    $K_\delta(\psi_{\mathbb{K}}^{\vec{\nu}}(a)) = \emptyset$ if $\alpha < \delta$. $K_\delta(\psi_{\mathbb{K}}^{\vec{\nu}}(a)) = \{a\} \cup \bigcup \{K_\delta(\gamma) : \gamma \in K(\nu)\}$ otherwise.

15. Let $\alpha = \psi_\pi^{\vec{\nu}}(a)$ with a strongly irreducible sequence $\vec{\nu} = (\nu_2, \ldots, \nu_{N-1})$ of ordinal terms in E, $a \in OT$, and $\pi \in OT \cap \mathbb{K}$ such that the following conditions are met:

    (a) $\forall \langle b, \rho, c \rangle \in K^2(\vec{\nu})(c \leq a \,\&\, K_\alpha(b) < b)$ and $K_\alpha(\pi) \cup K_\alpha(a) < a$.

    (b) There are a $k \, (2 \leq k \leq N - 2)$ and $a \geq b \in OT$ such that

$$\begin{aligned} & m_{k+1}(\pi) \neq 0, \nu_k = m_k(\pi) + \Lambda^{m_{k+1}(\pi)} \langle b, \pi, a \rangle, \\ & \forall i < k (\nu_i = m_i(\pi)), \forall i > k (\nu_i = 0) \end{aligned} \tag{6}$$

    Then $\alpha = \psi_\pi^{\vec{\nu}}(a) \in OT$, and $m_i(\alpha) = \nu_i$.

    $K_\delta(\psi_\pi^{\vec{\nu}}(a)) = \emptyset$ if $\alpha < \delta$. Otherwise $K_\delta(\psi_\pi^{\vec{\nu}}(a)) = \{a\} \cup K_\delta(a) \cup K_\delta(\pi) \cup \bigcup \{K_\delta(b) : \langle b, \rho, c \rangle \in K^2(\vec{\nu})\}$.

16. Let $\alpha = \psi_\pi^{\vec{\nu}}(a)$ with a strongly irreducible sequence $\vec{\nu} = (\nu_2, \ldots, \nu_{N-1}) \neq \vec{0}$ of ordinal terms in E, $a \in OT$, and $\pi \in OT \cap \mathbb{K}$ such that the following conditions are met:

    (a) $\forall \langle b, \rho, c \rangle \in K^2(\vec{\nu})(c \leq a \,\&\, K_\alpha(b) < b)$ and $K_\alpha(\pi) \cup K_\alpha(a) < a$.

(b) $\forall i > 2(m_i(\pi) = 0)$ and $(\nu_2, \ldots, \nu_{N-1}) <_{Ksl} m_2(\pi)$, cf. Definition 3.3.7.

(c)
$$\forall i(K_\pi(b_i) < a_i) \tag{7}$$
where $st(\nu_i) = \langle b_i, \rho_i, a_i \rangle$.

Then $\alpha = \psi_\pi^{\vec{\nu}}(a) \in OT$, and $m_i(\alpha) = \nu_i$.

$K_\delta(\psi_\pi^{\vec{\nu}}(a)) = \emptyset$ if $\alpha < \delta$. Otherwise $K_\delta(\psi_\pi^{\vec{\nu}}(a)) = \{a\} \cup K_\delta(a) \cup K_\delta(\pi) \cup \bigcup\{K_\delta(b) : \langle b, \rho, c \rangle \in K^2(\vec{\nu})\}$.

17. $\vec{m}(\alpha) := (m_i(\alpha))_i$.

Remark. Let us understand Definition 3.3 as follows. First a super set OT' together with a linear order $<'$ is defined. An element of OT' may be a well-formed term over symbols $\{0, \mathbb{K}, \Lambda, +, \omega, \varphi, \Omega, \psi\}$ without normal form conditions $=_{NF}$ and conditions in Definitions 3.3.13-16 such as $K_\alpha(\pi) \cup K_\alpha(a) < a$ for $\alpha = \psi_\pi(a)$. The linear order $<'$ on OT' is defined as in the proof of Lemma 3.6 below. After that a subset $OT \subset OT'$ is extracted with normal form conditions, and the restriction of $<'$ to OT results in a linear order $<$ on OT.

Proposition 3.4 *For any $\xi \in E$ and any $\langle b, \pi, a \rangle \in K(\xi)$, $b \leq a$ holds.*

Let $\psi_\pi^{\vec{\nu}}(a) \in OT$, and $\nu_i \in E$ be in the list $\vec{\nu}$. It is easy to see that indicators, i.e., the second and third components in $\nu_i = \langle \alpha, \mathbb{K}, a \rangle$ or in $\nu_i = \Lambda^{\xi_m}\langle b_m, \pi_m, a_m \rangle + \cdots + \Lambda^{\xi_0}\langle b_0, \pi_0, a_0 \rangle$ are uniquely determined from π, and when $\pi < \mathbb{K}$, from indicators of π together with numbers $(p_k)_{2 \leq k \leq N-1}$, i.e., from $\vec{\nu}$ without decorations implicit in Definition 3.3.16b, cf. Definition 3.3.7. For explicit calculations of indicators, see subsection 7.1.

Proposition 3.5 *For any $\alpha \in OT$ and any δ such that $\delta = 0, \mathbb{K}$ or $\delta = \psi_\pi^{\vec{\nu}}(b)$ for some $\pi, b, \vec{\nu}$, $\alpha \in \mathcal{H}_\gamma(\delta) \Leftrightarrow K_\delta(\alpha) < \gamma$.*

Proof. By induction on $\ell\alpha$. □

Lemma 3.6 *$(OT, <)$ is a computable notation system of ordinals. In particular the order type of the initial segment $\{\alpha \in OT : \alpha < \Omega_1\}$ is less than ω_1^{CK}.*

Specifically $\alpha < \beta$ is decidable for $\alpha, \beta \in OT$, and $\alpha \in OT$ is decidable for terms α over symbols $\{0, \mathbb{K}, \Lambda, +, \omega, \varphi, \Omega, \psi\}$.

Proof. These are shown simultaneously referring Propositions 2.17 and 3.5. Let us give recursive definitions only for terms $\Omega_\alpha, \psi_\kappa^{\vec{\nu}}(a) \in OT$.

First $\Omega_{\psi_\kappa^{\vec{\nu}}(a)} = \psi_\kappa^{\vec{\nu}}(a)$, i.e., $\Omega_\alpha < \psi_\kappa^{\vec{\nu}}(a) \Leftrightarrow \alpha < \psi_\kappa^{\vec{\nu}}(a)$, $\psi_\kappa^{\vec{\nu}}(a) < \Omega_\alpha \Leftrightarrow \psi_\kappa^{\vec{\nu}}(a) < \alpha$. Next $\Omega_\alpha < \psi_{\Omega_{\alpha+1}}(a) < \Omega_{\alpha+1}$.

Finally for $\psi_\pi^{\vec{\nu}}(b), \psi_\kappa^{\vec{\xi}}(a) \in OT$, $\psi_\pi^{\vec{\nu}}(b) < \psi_\kappa^{\vec{\xi}}(a)$ iff one of the following cases holds:

1. $\pi \leq \psi_\kappa^{\vec{\xi}}(a)$.

2. $b < a$, $\psi_{\pi}^{\vec{\nu}}(b) < \kappa$, $\forall\langle\gamma,\rho,c\rangle \in K^2(\vec{\nu})(K_{\psi_{\kappa}^{\vec{\xi}}(a)}(\gamma) < a)$ and $K_{\psi_{\kappa}^{\vec{\xi}}(a)}(\{\pi, b\}) < a$.

3. $b \geq a$, and $\exists\langle\delta,\rho,c\rangle \in K^2(\vec{\xi})(b \leq K_{\psi_{\pi}^{\vec{\nu}}(b)}(\delta)) \vee b \leq K_{\psi_{\pi}^{\vec{\nu}}(b)}(\{\kappa, a\})$.

4. $b = a$, $\pi = \kappa$, $\forall\langle\gamma,\rho,c\rangle \in K^2(\vec{\nu})(K_{\psi_{\kappa}^{\vec{\xi}}(a)}(\gamma) < a)$, and $\vec{\nu} <_{lx} \vec{\xi}$.

□

Proposition 3.7 *For $\alpha, \beta \in OT$, if $\alpha = \beta$, then the ordinal term α coincides with the ordinal term β.*

Proof. This is seen by induction on the sum $\ell\alpha + \ell\beta$ of lengths of ordinal terms $\alpha, \beta \in OT$. □

4 Operator controlled derivations

In this section, operator controlled derivations are first introduced, and in the next section inferences $(\mathbb{K} \in M_N)$ for Π_N-reflecting ordinals $\mathbb{K}$ are eliminated from operator controlled derivations of Σ_1-sentences φ^{L_Ω} over Ω.

In what follows except otherwise stated $\alpha, \beta, \gamma, \ldots, a, b, c, d, \ldots$ range over *ordinal terms* in OT, $\xi, \zeta, \nu, \mu, \iota, \ldots$ range over ordinal terms in E (with or without decorations), $\vec{\xi}, \vec{\zeta}, \vec{\nu}, \vec{\mu}, \vec{\iota}, \ldots$ range over finite sequences over ordinal terms in E, and $\pi, \kappa, \rho, \sigma, \tau, \lambda, \ldots$ range over regular ordinal terms, i.e., one of ordinal terms $\mathbb{K}$, $\Omega_{\beta+1}$, $\psi_{\pi}^{\vec{\nu}}(a)$ with $\vec{\nu} \neq \vec{0}$. *Reg* denotes the set of regular ordinal terms. In this and the next sections ordinal terms are decorated with indicators. We write $\alpha \in \mathcal{H}_a(\beta)$ for $K_\beta(\alpha) < a$.

4.1 Classes of sentences

Following Buchholz [11] let us introduce a language for ramified set theory RS.

Definition 4.1 *RS-terms* and their *levels* are inductively defined.

1. For each $\alpha \in OT \cap \mathbb{K}$, L_α is an RS-term of level α.

2. If $\phi(x, y_1, \ldots, y_n)$ is a set-theoretic formula in the language $\{\in\}$, and $a_1, \ldots, a_n$ are RS-terms of levels$< \alpha$, then $[x \in L_\alpha : \phi^{L_\alpha}(x, a_1, \ldots, a_n)]$ is an RS-term of level α.

Each ordinal term α is denoted by the ordinal term $[x \in L_\alpha : x \text{ is an ordinal}]$, whose level is α.

Definition 4.2 *1.* $|a|$ denotes the level of RS-terms a, and $Tm(\alpha)$ the set of RS-terms of level$< \alpha$. $Tm = Tm(\mathbb{K})$ is then the set of RS-terms, which are denoted by $a, b, c, d, \ldots$

2. *RS-formulas* are constructed from *literals* $a \in b, a \notin b$ by propositional connectives $\vee, \wedge$, bounded quantifiers $\exists x \in a, \forall x \in a$ and unbounded quantifiers $\exists x, \forall x$. Unbounded quantifiers $\exists x, \forall x$ are denoted by $\exists x \in L_{\mathbb{K}}, \forall x \in L_{\mathbb{K}}$, resp.

3. For RS-terms and RS-formulas ι, $\mathsf{k}(\iota)$ denotes the set of ordinal terms α such that the constant L_α occurs in ι.

4. For a set-theoretic Σ_n-formula $\psi(x_1, \ldots, x_m)$ in $\{\in\}$ and $a_1, \ldots, a_m \in Tm(\kappa)$, $\psi^{L_\kappa}(a_1, \ldots, a_m)$ is a $\Sigma_n(\kappa)$-*formula*, where $n = 0, 1, 2, \ldots$ and $\kappa \leq \mathbb{K}$. $\Pi_n(\kappa)$-formulas are defined dually.

5. For $\theta \equiv \psi^{L_\kappa}(a_1, \ldots, a_m) \in \Sigma_n(\kappa)$ and $\lambda < \kappa$, $\theta^{(\lambda,\kappa)} :\equiv \psi^{L_\lambda}(a_1, \ldots, a_m)$.

Note that the level $|t| = \max(\{0\} \cup \mathsf{k}(t))$ for RS-terms t. In what follows we need to consider *sentences*. Sentences are denoted A, C possibly with indices.

The assignment of disjunctions and conjunctions to sentences is defined as in [11].

Definition 4.3 *1.* For $b, a \in Tm(\mathbb{K})$ with $|b| < |a|$,

$$b\varepsilon a :\equiv \begin{cases} A(b) & a \equiv [x \in L_\alpha : A(x)] \\ b \notin L_0 & a \equiv L_\alpha \end{cases}$$

and $a = b :\equiv (\forall x \in a(x \in b) \wedge \forall x \in b(x \in a))$.

2. For $b, a \in Tm(\mathbb{K})$ and $J := Tm(|a|)$

$$b \in a :\simeq \bigvee (c\varepsilon a \wedge c = b)_{c \in J} \text{ and } b \notin a :\simeq \bigwedge (c \not\varepsilon a \vee c \neq b)_{c \in J}$$

3. $(A_0 \vee A_1) :\simeq \bigvee(A_\iota)_{\iota \in J}$ and $(A_0 \wedge A_1) :\simeq \bigwedge(A_\iota)_{\iota \in J}$ for $J := 2$.

4. For $a \in Tm(\mathbb{K}) \cup \{L_{\mathbb{K}}\}$ and $J := Tm(|a|)$

$$\exists x \in a\, A(x) :\simeq \bigvee (b\varepsilon a \wedge A(b))_{b \in J} \text{ and } \forall x \in a\, A(x) :\simeq \bigwedge (b \not\varepsilon a \vee A(b))_{b \in J}.$$

The rank $\mathrm{rk}(\iota)$ of sentences or terms ι is defined as in [11].

Definition 4.4 *1.* $\mathrm{rk}(\neg A) := \mathrm{rk}(A)$.

2. $\mathrm{rk}(L_\alpha) = \omega\alpha$.

3. $\mathrm{rk}([x \in L_\alpha : A(x)]) = \max\{\omega\alpha + 1, \mathrm{rk}(A(L_0)) + 2\}$.

4. $\mathrm{rk}(a \in b) = \max\{\mathrm{rk}(a) + 6, \mathrm{rk}(b) + 1\}$.

5. $\mathrm{rk}(A_0 \vee A_1) := \max\{\mathrm{rk}(A_0), \mathrm{rk}(A_1)\} + 1$.

6. $\mathrm{rk}(\exists x \in a\, A(x)) := \max\{\omega \mathrm{rk}(a), \mathrm{rk}(A(L_0)) + 2\}$ for $a \in Tm(\mathbb{K}) \cup \{L_{\mathbb{K}}\}$.

Proposition 4.5 *Let A be a sentence with $A \simeq \bigvee(A_\iota)_{\iota \in J}$ or $A \simeq \bigwedge(A_\iota)_{\iota \in J}$.*

1. $\mathrm{rk}(A) < \mathbb{K} + \omega$.

2. $|A| \leq \mathrm{rk}(A) \in \{\omega|A| + i : i \in \omega\}$.

3. $\forall\iota \in J(\mathrm{rk}(A_\iota) < \mathrm{rk}(A))$.

4. $\mathrm{rk}(A) < \lambda \Rightarrow A \in \Sigma_0(\lambda)$

4.2 Operator controlled derivations

By an *operator* we mean a map $\mathcal{H}$, $\mathcal{H} : \mathcal{P}(OT) \to \mathcal{P}(OT)$, such that

1. $\forall X \subset OT[X \subset \mathcal{H}(X)]$.

2. $\forall X, Y \subset OT[Y \subset \mathcal{H}(X) \Rightarrow \mathcal{H}(Y) \subset \mathcal{H}(X)]$.

For an operator $\mathcal{H}$ and $\Theta, \Theta_1 \subset OT$, $\mathcal{H}[\Theta](X) := \mathcal{H}(X \cup \Theta)$, and $\mathcal{H}[\Theta][\Theta_1] := (\mathcal{H}[\Theta])[\Theta_1]$, i.e., $\mathcal{H}[\Theta][\Theta_1](X) = \mathcal{H}(X \cup \Theta \cup \Theta_1)$.

Obviously $\mathcal{H}_\alpha$ is an operator for any α, and if $\mathcal{H}$ is an operator, then so is $\mathcal{H}[\Theta]$.

Sequents are finite sets of sentences, and inference rules are formulated in one-sided sequent calculus. Let $\mathcal{H} = \mathcal{H}_\gamma$ ($\gamma \in OT$) be an operator, Θ a finite set of $\mathbb{K}$, Γ a sequent, $a \in OT$ and $b \in OT \cap (\mathbb{K} + \omega)$.

We define a relation $(\mathcal{H}_\gamma, \Theta) \vdash^a_b \Gamma$, which is read 'there exists an infinitary derivation of Γ which is Θ-*controlled* by $\mathcal{H}_\gamma$, and whose height is at most a and its cut rank is less than b'.

$\kappa, \lambda, \sigma, \tau, \pi$ ranges over regular ordinal terms.

Definition 4.6 $(\mathcal{H}_\gamma, \Theta) \vdash^a_b \Gamma$ holds if

$$\mathsf{k}(\Gamma) \cup \{a\} \subset \mathcal{H}_\gamma[\Theta] \tag{8}$$

and one of the following cases holds:

($\bigvee$) $A \simeq \bigvee\{A_\iota : \iota \in J\}$, $A \in \Gamma$ and there exist $\iota \in J$ and $a(\iota) < a$ such that

$$|\iota| < a \tag{9}$$

and $(\mathcal{H}_\gamma, \Theta) \vdash^{a(\iota)}_b \Gamma, A_\iota$.

($\bigwedge$) $A \simeq \bigwedge\{A_\iota : \iota \in J\}$, $A \in \Gamma$ and for every $\iota \in J$ there exists an $a(\iota) < a$ such that $(\mathcal{H}_\gamma, \Theta \cup \{\mathsf{k}(\iota)\}) \vdash^{a(\iota)}_b \Gamma, A_\iota$.

(*cut*) There exist $a_0 < a$ and C such that $\mathrm{rk}(C) < b$ and $(\mathcal{H}_\gamma, \Theta) \vdash^{a_0}_b \Gamma, \neg C$ and $(\mathcal{H}_\gamma, \Theta) \vdash^{a_0}_b C, \Gamma$.

($\Omega \in M_2$) There exist ordinals a_ℓ, $a_r(\alpha)$ and a sentence $C \in \Pi_2(\Omega)$ such that $\sup\{a_\ell + 1, a_r(\alpha) + 1 : \alpha < \Omega\} \leq a$, $b \geq \Omega$, $(\mathcal{H}_\gamma, \Theta) \vdash^{a_\ell}_b \Gamma, C$ and $(\mathcal{H}_\gamma, \Theta \cup \{\omega\alpha\}) \vdash^{a_r(\alpha)}_b \neg C^{(\alpha,\Omega)}, \Gamma$ for any $\alpha < \Omega$.

$(\pi \in Mh_2(\vec{\xi}), k, \vec{\nu})$ There exist a regular ordinal term $\pi \in \mathcal{H}_\gamma[\Theta] \cap (b+1)$, a positive integer $2 \leq k \leq N$, where $\pi = \mathbb{K} \Leftrightarrow k = N$, and in this case $\vec{\xi} = \vec{0}$ with $lh(\vec{\xi}) = N-1$, $\vec{\nu} = \vec{0}$. For the case $\pi < \mathbb{K}$, let $\vec{\xi} = (\xi_2, \ldots, \xi_{N-1}) = \vec{m}(\pi)$. Also there is a strongly irreducible sequence $\vec{\nu} = (\nu_2, \ldots, \nu_{N-1})$ of ordinals $\nu_i \in E$.

Moreover there are ordinals $a_\ell, a_r(\rho), a_0$, and a finite set Δ of $\Sigma_k(\pi)$-sentences enjoying the following conditions :

1. When $\pi < \mathbb{K}$, the following two conditions hold:

$$\forall i < k(\nu_i \leq_{pt} \xi_i) \,\&\, (\nu_k, \ldots, \nu_{N-1}) <_{Ksl} \xi_k \tag{10}$$

and

$$\forall \langle d, \rho, c \rangle \in K^2(\vec{\nu})(c \leq \gamma \,\&\, d \in \mathcal{H}_c(\Theta \cap \pi)) \tag{11}$$

2. For each $\delta \in \Delta$

$$(\mathcal{H}_\gamma, \Theta) \vdash^{a_\ell}_b \Gamma, \neg\delta$$

3. Let $H_k(\vec{\nu}, \pi, \gamma, \Theta)$ denote the *resolvent class* for $\pi \in Mh_2(\vec{\xi})$ with respect to k, $\vec{\nu}$, γ and Θ:

$$H_k(\vec{\nu}, \pi, \gamma, \Theta) := \{\rho \in Mh_2(\vec{\nu}) \cap \pi : \mathcal{H}_\gamma(\rho) \cap \pi \subset \rho \,\&\, \Theta \cap \pi \subset \rho\}$$

where

$$\rho \in Mh_2(\vec{\nu}) :\Leftrightarrow \vec{\nu} \leq_{pt} \vec{m}(\rho) :\Leftrightarrow \forall i(\nu_i \leq_{pt} m_i(\rho))$$

In the case $\pi = \mathbb{K}$, i.e., if $k = N$, $\rho \in Mh_2(\vec{0}) :\Leftrightarrow \rho \in Reg$, i.e.,

$$H_N(\vec{0}, \mathbb{K}, \gamma, \Theta) := \{\rho \in Reg \cap \mathbb{K} : \mathcal{H}_\gamma(\rho) \cap \mathbb{K} \subset \rho \,\&\, \Theta \cap \mathbb{K} \subset \rho\}$$

Then for any $\rho \in H_k(\vec{\nu}, \pi, \gamma, \Theta)$

$$(\mathcal{H}_\gamma, \Theta \cup \{\rho\} \cup \{(\rho, \vec{\nu})\}) \vdash^{a_r(\rho)}_b \Gamma, \Delta^{(\rho,\pi)}$$

4.
$$\sup\{a_\ell, a_r(\rho) : \rho \in H_k(\vec{\nu}, \pi, \gamma, \Theta)\} \leq a_0 \in \mathcal{H}_\gamma[\Theta] \cap a \tag{12}$$

We will state some lemmas for the operator controlled derivations with sketches of their proofs since these can be shown as in [11]. In what follows by an operator $\mathcal{H}$ we mean an $\mathcal{H}_\gamma$ for an ordinal γ.

Lemma 4.7 *Let* $(\mathcal{H}_\gamma, \Theta) \vdash^a_b \Gamma$.

1. $(\mathcal{H}_{\gamma'}, \Theta \cup \Theta_0) \vdash^{a'}_{b'} \Gamma, \Delta$ *for any* $\gamma' \geq \gamma$, *any* Θ_0, *and any* $a' \geq a$, $b' \geq b$ *such that* $\mathsf{k}(\Delta) \cup \{a'\} \subset \mathcal{H}_{\gamma'}[\Theta \cup \Theta_0]$.

2. *Assume* $\Theta_1 \cup \{c\} = \Theta$, $c \in \mathcal{H}_\gamma[\Theta_1]$. *Then* $(\mathcal{H}_\gamma, \Theta_1) \vdash^a_b \Gamma$.

Lemma 4.8 (Tautology) $(\mathcal{H}, \mathsf{k}(\Gamma\cup\{A\}))\vdash_0^{2\mathrm{rk}(A)} \Gamma, \neg A, A$.

Lemma 4.9 (Inversion)
Let $A \simeq \bigwedge(A_\iota)_{\iota\in J}$, and $(\mathcal{H},\Theta)\vdash_b^a \Gamma$ with $A\in\Gamma$. Then for any $\iota\in J$, $(\mathcal{H},\Theta\cup\mathsf{k}(\iota))\vdash_b^a \Gamma, A_\iota$ holds.

Lemma 4.10 (Boundedness)
Suppose $(\mathcal{H},\Theta)\vdash_c^a \Gamma, C$ for a $C\in\Sigma_1(\lambda)$, and *$a\leq b\in\mathcal{H}\cap\lambda$. Then $(\mathcal{H},\Theta)\vdash_c^a \Gamma, C^{(b,\lambda)}$.*

Lemma 4.11 (Persistency)
Suppose $(\mathcal{H},\Theta)\vdash_c^a \Gamma, C^{(b,\lambda)}$ for a $C\in\Sigma_1(\lambda)$ and a $b<\lambda\in\mathcal{H}[\Theta]$. Then $(\mathcal{H},\Theta)\vdash_c^a \Gamma, C$.

Lemma 4.12 (Predicative Cut-elimination)
Suppose $(\mathcal{H},\Theta)\vdash_{c+\omega^a}^b \Gamma$, $a\in\mathcal{H}[\Theta]$ and $]c, c+\omega^a]\cap Reg=\emptyset$. Then $(\mathcal{H},\Theta)\vdash_c^{\varphi ab} \Gamma$.

Lemma 4.13 (Embedding of Axioms)
For each axiom A in $\mathsf{KP}\Pi_N$, there is an $m<\omega$ such that for any operator $\mathcal{H}=\mathcal{H}_\gamma$, $(\mathcal{H},\emptyset)\vdash_{\mathbb{K}+m}^{\mathbb{K}\cdot 2} A$ holds.

Lemma 4.14 (Embedding)
If $\mathsf{KP}\Pi_N\vdash\Gamma$ for sets Γ of sentences, there are $m,k<\omega$ such that for any operator $\mathcal{H}=\mathcal{H}_\gamma$, $(\mathcal{H},\emptyset)\vdash_{\mathbb{K}+m}^{\mathbb{K}\cdot 2+k} \Gamma$ holds

5 Lowering and eliminating higher Mahlo operations

In the section we eliminate inferences $(\mathbb{K}\in M_N)$ for Π_N-reflection.

In the following Lemmas 5.1 and 5.3, for ordinal terms ρ, a and sequences $\vec{\nu}$ including the case $\vec{\nu}=\vec{0}$, a term $\psi_\rho^{\vec{\nu}}(a)$ may not in OT, i.e., $\psi_\rho^{\vec{\nu}}(a)\in OT'$, cf. **Remark** after Definition 3.3.

$\alpha\#\beta$ denotes the natural sum of ordinal terms α,β.

Lemma 5.1 *Suppose for ordinal terms $\gamma, a, \pi\in OT$*

$$(\mathcal{H}_\gamma,\Theta)\vdash_\pi^a \Gamma$$

where $\{\gamma,\pi\}\subset\mathcal{H}_\gamma[\Theta]$, $\Theta\subset\pi$ and $\Gamma\subset\Pi_{k+1}(\pi)$ for some $2\leq k\leq N-1$.

Let $\vec{\xi}=\vec{0}$ with $lh(\vec{\xi})=N-1$ when $\pi=\mathbb{K}\,\&\,k=N-1$. When $k<N-1$, assume $\pi<\mathbb{K}$, and let $\vec{m}(\pi)=\vec{\xi}=(\xi_2,\ldots,\xi_{k+1})*\vec{0}$, and assume $\xi_{k+1}\neq 0$ and

$$\forall\langle b,\rho,c\rangle\in K^2(\vec{\xi})(b\in\mathcal{H}_\gamma[\Theta]\,\&\,c\leq\gamma) \tag{13}$$

For an ordinal term $\gamma+a\leq c<\Lambda$, let us define a sequence $\vec{\zeta^c}(a):=(\zeta_2^c(a),\ldots,\zeta_k^c(a))*\vec{0}$ by $\vec{\zeta^c}(a)=\vec{0}*(\langle\gamma+a,\mathbb{K},c\rangle)$ with $lh(\vec{\zeta^c}(a))=N-2$ when $\pi=\mathbb{K}$. Otherwise $\zeta_k^c(a)=\xi_k+\Lambda^{\xi_{k+1}}\langle\gamma+a,\pi,c\rangle$ and $\zeta_i^c(a)=\xi_i$ for $i<k$.

Let $\kappa < \pi$ be an ordinal term such that

$$\vec{\zeta^c}(a) \leq^*_k \vec{m}(\kappa) :\Leftrightarrow \exists \mu(\zeta^c_k(a) \leq_{Kst} \mu \leq_{pt} m_k(\kappa)) \,\&\, \forall i < k(\zeta^c_i(a) \leq_{pt} m_i(\kappa))$$

$\mathcal{H}_\gamma(\kappa) \cap \pi \subset \kappa$ and

$$\Theta \subset \kappa \tag{14}$$

Let $\gamma(a,b) = \gamma\#a\#b$, $\beta(a,b) = \psi_\pi(\gamma(a,b))$, and $c_1 = \max\{\gamma(a,\kappa)+1, c\}$. Then the following holds:

$$(\mathcal{H}_{c_1}, \Theta \cup \{\kappa\}) \vdash^{\beta(a,\kappa)}_{\kappa} \Gamma^{(\kappa,\pi)} \tag{15}$$

Proof by induction on a. Let $\kappa < \pi$, $\vec{\zeta^c}(a) \leq^*_k \vec{m}(\kappa)$, $\mathcal{H}_\gamma(\kappa) \cap \pi \subset \kappa$ and $\Theta \subset \kappa$.

We see that $\vec{\zeta^c}(a)$ is strongly irreducible from Proposition 3.2. We see from (8) and (14) that

$$\mathsf{k}(\Gamma) \cap \pi \subset \mathcal{H}_\gamma(\kappa) \cap \pi \subset \kappa \tag{16}$$

For any $a \in \mathcal{H}_\gamma[\Theta]$, we have $\{\gamma, \pi, a, \kappa\} \subset \mathcal{H}_\gamma(\pi)$ by $\Theta \cup \{\kappa\} \subset \pi$. Hence for $\gamma(a,\kappa) = \gamma\#a\#\kappa$, $\{\gamma(a,\kappa), \pi\} \subset \mathcal{H}_\gamma(\pi)$, and $\{\gamma(a,\kappa), \pi\} \subset \mathcal{H}_{\gamma(a,\kappa)}(\beta(a,\kappa))$ by the definition (4). Therefore $\kappa \in \mathcal{H}_{\gamma(a,\kappa)}(\beta(a,\kappa)) \cap \pi \subset \beta(a,\kappa)$ by Proposition 2.4, and $\Theta \subset \beta(a,\kappa) < \pi$. Thus we obtain

$$\{a_0, a_1\} \subset \mathcal{H}_\gamma[\Theta \cup \Theta_0] \,\&\, a_0 < a_1 \,\&\, \Theta_0 \subset \kappa \Rightarrow \beta(a_0,\kappa) < \beta(a_1,\kappa)$$

Likewise we see from $\Theta \subset \pi < \psi_{\pi^+}(\gamma + a)$ that $\psi_{\pi^+}(\gamma + a_0) < \psi_{\pi^+}(\gamma + a_1)$, i.e.,

$$\{a_0, a_1\} \subset \mathcal{H}_\gamma[\Theta \cup \Theta_0] \,\&\, a_0 < a_1 \,\&\, \Theta_0 \subset \pi \Rightarrow \zeta^c_k(a_0) <_{Kst} \zeta^c_k(a) \,\&\, \vec{\zeta^c}(a_0) \leq^*_k \vec{m}(\kappa) \tag{17}$$

Case 1. First consider the case when the last inference is a $(\pi \in Mh_2(\vec{\xi}), k+1, \vec{\nu})$.

We have $a_\ell \in \mathcal{H}_\gamma[\Theta] \cap a$, and $a_r(\rho) \in \mathcal{H}_\gamma[\Theta \cup \{\rho\}] \cap a$. Δ is a finite set of $\Sigma_{k+1}(\pi)$-sentences.

We have for each $\delta \in \Delta$

$$(\mathcal{H}_\gamma, \Theta) \vdash^{a_\ell}_{\pi} \Gamma, \neg\delta \tag{18}$$

and for each $\rho \in H_{k+1}(\vec{\nu}, \pi, \gamma, \Theta)$

$$(\mathcal{H}_\gamma, \Theta \cup \{\rho\}) \vdash^{a_r(\rho)}_{\pi} \Gamma, \Delta^{(\rho,\pi)} \tag{19}$$

where $H_{k+1}(\vec{\nu}, \pi, \gamma, \Theta)$ is the resolvent class for $\pi \in Mh_2(\vec{\xi})$ with respect to $k+1$, $\vec{\nu}$, γ and Θ:

$$H_{k+1}(\vec{\nu}, \pi, \gamma, \Theta) := \{\rho \in Mh_2(\vec{\nu}) \cap \pi : \mathcal{H}_\gamma(\rho) \cap \pi \subset \rho \,\&\, \Theta \cap \pi \subset \rho\}$$

When $\pi < \mathbb{K}$, $\vec{\nu} = (\nu_2, \ldots, \nu_{N-1})$ is a strongly irreducible sequence such that $\forall \langle d, \rho, c\rangle \in K^2(\vec{\nu})(c \leq \gamma \,\&\, d \in \mathcal{H}_c(\Theta \cap \pi))$, $\forall i < k+1(\nu_i \leq_{pt} \xi_i)$, and $(\nu_{k+1}, \ldots, \nu_{N-1}) <_{Ksl} \xi_{k+1}$, cf. (11) and (10).

Let $\Gamma_0 = \Gamma \cap \Sigma_k(\pi)$ and $\{\forall x \in L_\pi\, \theta_i(x) : i = 1, \ldots, n\}\, (n \geq 0) = \Gamma \setminus \Gamma_0$ for $\Sigma_k(\pi)$-formulas $\theta_i(x)$. Let us fix $\vec{d} = \{d_1, \ldots, d_n\} \subset Tm(\kappa)$ arbitrarily. Put $\mathsf{k}(\vec{d}) = \bigcup\{\mathsf{k}(d_i) : i = 1, \ldots, n\}$ and $\Gamma(\vec{d}) = \Gamma_0 \cup \{\theta_i(d_i) : i = 1, \ldots, n\}$.

By Inversion lemma 4.9 from (18) we obtain for each $\delta \in \Delta$

$$(\mathcal{H}_\gamma, \Theta \cup \mathsf{k}(\vec{d})) \vdash^{a_\ell}_\pi \Gamma(\vec{d}), \neg\delta \tag{20}$$

Let

$$H_{k+1}(\vec{\nu}, \kappa, c_1, \Theta \cup \{\kappa\} \cup \mathsf{k}(\vec{d})) =$$
$$\{\rho \in Mh_2(\vec{\nu}) \cap \kappa : \mathcal{H}_{c_1}(\rho) \cap \kappa \subset \rho \,\&\, (\Theta \cup \{\kappa\} \cup \mathsf{k}(\vec{d})) \cap \kappa \subset \rho\}$$

Then $\mathsf{k}(\vec{d}) < \rho$ for $\rho \in H_{k+1}(\vec{\nu}, \kappa, c_1, \Theta \cup \{\kappa\} \cup \mathsf{k}(\vec{d}))$. By $\Theta \cap \pi \subset \mathcal{H}_\gamma(\kappa) \cap \pi \subset \kappa$ and $\gamma \leq c_1$ we have

$$\rho \in H_{k+1}(\vec{\nu}, \kappa, c_1, \Theta \cup \{\kappa\} \cup \mathsf{k}(\vec{d})) \Rightarrow \rho \in H_{k+1}(\vec{\nu}, \pi, \gamma, \Theta) \tag{21}$$

If $\rho \in H_{k+1}(\vec{\nu}, \kappa, c_1, \Theta \cup \{\kappa\} \cup \mathsf{k}(\vec{d}))$, then $\rho < \kappa$ for (14). For each $\rho \in H_{k+1}(\vec{\nu}, \kappa, c_1, \Theta \cup \{\kappa\} \cup \mathsf{k}(\vec{d}))$, IH with (19), (21) and (17) yields for $c_1 \geq \gamma(a_r(\rho), \kappa) + 1$

$$(\mathcal{H}_{c_1}, \Theta \cup \mathsf{k}(\vec{d}) \cup \{\rho, \kappa\}) \vdash^{\beta(a_r(\rho), \kappa)}_\kappa \Gamma^{(\kappa,\pi)}, \Delta^{(\rho,\pi)} \tag{22}$$

Next let $\rho \in Mh_2(\vec{\zeta}^c(a_\ell)) \cap H_{k+1}(\vec{\nu}, \kappa, c_1, \Theta \cup \{\kappa\} \cup \mathsf{k}(\vec{d}))$. Then $\Theta \cup \mathsf{k}(\vec{d}) \subset \rho$ for (14), and $\vec{\zeta}^c(a_\ell) \leq^*_k \vec{m}(\rho)$ by $\vec{\zeta}^c(a_\ell) \leq_{pt} \vec{m}(\rho)$. For any $\rho \in Mh_2(\vec{\zeta}(a_\ell)) \cap H_{k+1}(\vec{\nu}, \kappa, c_1, \Theta \cup \{\kappa\} \cup \mathsf{k}(\vec{d}))$ and for any $\delta \in \Delta$, IH with (20) yields for $c_1 \geq \gamma(a_\ell, \rho) + 1$

$$(\mathcal{H}_{c_1}, \Theta \cup \mathsf{k}(\vec{d}) \cup \{\rho, \kappa\}) \vdash^{\beta(a_\ell, \rho)}_\rho \Gamma(\vec{d})^{(\rho,\pi)}, \neg\delta^{(\rho,\pi)} \tag{23}$$

Now let

$$M_\ell := Mh_2(\vec{\zeta}(a_\ell)) \cap H_{k+1}(\vec{\nu}, \kappa, c_1, \Theta \cup \{\kappa\} \cup \mathsf{k}(\vec{d}))$$

From (22) and (23) by several (cut)'s of $\delta^{(\rho,\pi)}$ with $\mathrm{rk}(\delta^{(\rho,\pi)}) < \kappa$ we obtain for $a(\rho) = \max\{a_\ell, a_r(\rho)\}$ and some $p < \omega$

$$\{(\mathcal{H}_{c_1}, \Theta \cup \mathsf{k}(\vec{d}) \cup \{\kappa, \rho\}\}) \vdash^{\beta(a(\rho), \kappa) + p}_\kappa \Gamma(\vec{d})^{(\rho,\pi)}, \Gamma^{(\kappa,\pi)} : \rho \in M_\ell\} \tag{24}$$

On the other hand we have by Tautology lemma 4.8 for each $\theta(\vec{d})^{(\kappa,\pi)} \in \Gamma(\vec{d})^{(\kappa,\pi)}$

$$(\mathcal{H}_\gamma, \Theta \cup \mathsf{k}(\vec{d}) \cup \{\kappa\}) \vdash^{2\mathrm{rk}(\theta(\vec{d})^{(\kappa,\pi)})}_0 \Gamma(\vec{d})^{(\kappa,\pi)}, \neg\theta(\vec{d})^{(\kappa,\pi)} \tag{25}$$

where $2\mathrm{rk}(\theta(\vec{d})^{(\kappa,\pi)}) \leq \kappa + p$ for some $p < \omega$.

Moreover we have $\sup\{2\mathrm{rk}(\theta(\vec{d})^{(\kappa,\pi)}), \beta(a(\rho), \kappa) + p : \rho \in M_\ell\} \leq \beta(a_0, \kappa) + p \in \mathcal{H}_\gamma[\Theta \cup \{\kappa\}]$, where $\sup\{a_\ell, a_r(\rho) : \rho \in H_{k+1}(\vec{\nu}, \pi, \gamma, \Theta)\} \leq a_0 < a$ by (12).

Let $\vec{\mu} = (\mu_2, \ldots, \mu_{N-1}) = \max\{\vec{\zeta^c}(a_\ell), \vec{\nu}\}$ with

$$\mu_i = \max\{\zeta^c_i(a_\ell), \nu_i\} = \begin{cases} \zeta^c_i(a_\ell) & i \leq k \\ \nu_i & i > k \end{cases}$$

since $\nu_i \leq_{pt} \xi_i \leq_{pt} \zeta^c_i(a_\ell)$ for $i < k+1$.

Claim 5.2

$$M_\ell = \{\rho \in Mh_2(\vec{\mu}) \cap \kappa : \mathcal{H}_{c_1}(\rho) \cap \kappa \subset \rho \,\&\, (\Theta \cup \{\kappa\} \cup \mathsf{k}(\vec{d})) \cap \kappa \subset \rho\}$$

and M_ℓ is the resolvent class $H_k(\vec{\mu}, \kappa, c_1, \Theta \cup \{\kappa\} \cup \mathsf{k}(\vec{d}))$ for $\kappa \in Mh_2(\vec{m}(\kappa))$ with respect to k, $\vec{\mu}$, c_1 and $\Theta \cup \{\kappa\} \cup \mathsf{k}(\vec{d})$.

Proof of Claim 5.2.

First the fact that $\rho \in Mh_2(\vec{\mu}) \Leftrightarrow \rho \in Mh_2(\vec{\zeta^c}(a_\ell)) \cap Mh_2(\vec{\nu})$ is seen from $\vec{\nu}, \vec{\zeta^c}(a_\ell) \leq_{pt} \vec{\mu}$, i.e., $\nu_i \leq_{pt} \xi_i \leq_{pt} \zeta^c_i(a_\ell) = \mu_i$ for $i \leq k$ and $\zeta^c_i(a_\ell) = 0 \leq_{pt} \nu_i = \mu_i$ for $i > k$.

When $\pi = \mathbb{K}$, $\vec{\mu} = \vec{0} * (\langle \gamma + a, \mathbb{K}, c \rangle)$. In what follows let $\pi < \mathbb{K}$.

Consider the first half of (11), which say that $\forall \langle d, \rho, c_0 \rangle \in K^2(\vec{\mu})(c_0 \leq c_1)$. This is seen from (13) and (11) on the inference $(\pi \in Mh_2(\vec{\xi}), k+1, \vec{\nu})$, and $\gamma + a \leq c_1$.

Next consider the second half of (11). We have $d \in \mathcal{H}_{c_0}[\Theta] \subset \mathcal{H}_{c_0}[\Theta']$ for any $\langle d, \rho, c_0 \rangle \in K^2(\vec{\xi}) \cup K^2(\vec{\zeta^c}(a_\ell))$ with $\Theta \cap \pi = \Theta \subset \Theta' = (\Theta \cup \{\kappa\} \cup \mathsf{k}(\vec{d})) \cap \kappa$. Hence $\forall \langle d, \rho, c_0 \rangle \in K^2(\vec{\mu})(d \in \mathcal{H}_{c_0}(\Theta'))$.

Finally consider the condition (10). $\forall i < k(\mu_i = \zeta^c_i(a_\ell) = \xi_i = \zeta^c_i(a) \leq_{pt} m_i(\kappa))$, and for some λ $\mu_k = \zeta^c_k(a_\ell) <_{Kst} \zeta^c_k(a) \leq_{Kst} \lambda \leq_{pt} m_k(\kappa)$ by (17). On the other hand we have $(\mu_{k+1}, \ldots, \mu_{N-1}) = (\nu_{k+1}, \ldots, \nu_{N-1}) <_{Ksl} \xi_{k+1} = te(\zeta^c_k(a)) = te(\lambda)$. Hence we obtain $(\mu_k, \mu_{k+1}, \ldots, \mu_{N-1}) <_{Ksl} m_k(\kappa)$. Thus Claim 5.2 is shown. □

Since $\neg\Gamma(\vec{d})^{(\kappa,\pi)}$ consists of $\Pi_k(\kappa)$-sentences, by an inference rule $(\kappa \in Mh_2(\vec{m}(\kappa)), k, \vec{\mu})$ with its resolvent class M_ℓ, we conclude from (25) and (24) that

$$(\mathcal{H}_{c_1}, \Theta \cup \{\kappa\} \cup \mathsf{k}(\vec{d})) \vdash^{\beta(a_0,\kappa)+p+1}_{\kappa} \Gamma(\vec{d})^{(\kappa,\pi)}, \Gamma^{(\kappa,\pi)}$$

Since $\vec{d} \subset Tm(\kappa)$ is arbitrary, several $(\bigwedge)$'s yield (15).

Case 2. Second consider the case when the last inference is a $(\pi \in Mh_2(\vec{\xi}), j, \vec{\nu})$ for a $j < k+1$. Δ is a finite set of $\Sigma_j(\pi)$-sentences.

We have for each $\delta \in \Delta$

$$(\mathcal{H}_\gamma, \Theta) \vdash^{a_\ell}_{\pi} \Gamma, \neg\delta$$

where $a_\ell \in \mathcal{H}_\gamma[\Theta] \cap a$.

Also we have for each $\rho \in H_j(\vec{\nu}, \pi, \gamma, \Theta)$

$$(\mathcal{H}_\gamma, \Theta \cup \{\rho\}) \vdash^{a_r(\rho)}_{\pi} \Gamma, \Delta^{(\rho,\pi)}$$

where $a_r(\rho) \in \mathcal{H}_\gamma[\Theta \cup \{\rho\}] \cap a$.

$\vec{\nu}$ is a sequence such that $\forall i < j(\nu_i \leq_{pt} \xi_i)$ and $(\nu_j, \ldots, \nu_{N-1}) <_{Ksl} \xi_j$ for $\vec{\xi} = \vec{m}(\pi)$. We have $\forall i < j(\nu_i \leq_{pt} \xi_i \leq_{pt} \zeta_i^c(a) \leq_{pt} m_i(\kappa))$. If $j < k$, then $(\nu_j, \ldots, \nu_{N-1}) <_{Ksl} \zeta_j^c(a) \leq_{pt} m_j(\kappa)$ yields $(\nu_j, \ldots, \nu_{N-1}) <_{Ksl} m_j(\kappa)$. Let $j = k$. There exists a λ such that $\zeta_k^c(a) \leq_{Kst} \lambda \leq_{pt} m_k(\kappa)$. Then $te(\zeta_k^c(a)) = te(\lambda)$ and $\forall \mu <_{pt} \zeta_k^c(a)(\mu <_{pt} \lambda)$. Hence $(\nu_k, \ldots, \nu_{N-1}) <_{Ksl} \zeta_k^c(a) \leq_{Kst} \lambda \leq_{pt} m_k(\kappa)$ yields $(\nu_k, \ldots, \nu_{N-1}) <_{Ksl} m_k(\kappa)$.

Let $H_j(\vec{\nu}, \kappa, c_1, \Theta \cup \{\kappa\})$ be the resolvent class for $\kappa \in Mh_2(\vec{m}(\kappa))$ with respect to j, $\vec{\nu}$, c_1 and $\Theta \cup \{\kappa\}$. We have $H_j(\vec{\nu}, \kappa, c_1, \Theta \cup \{\kappa\}) \subset H_j(\vec{\nu}, \pi, \gamma, \Theta)$.

By IH we have for each $\delta \in \Delta$

$$(\mathcal{H}_{c_1}, \Theta \cup \{\kappa\}) \vdash_{\kappa}^{\beta(a_\ell, \kappa)} \Gamma^{(\kappa,\pi)}, \neg\delta^{(\kappa,\pi)}$$

and for each $\rho \in H_j(\vec{\nu}, \kappa, c_1, \Theta \cup \{\kappa\})$

$$(\mathcal{H}_{c_1}, \Theta \cup \{\kappa, \rho\}) \vdash_{\kappa}^{\beta(a_r(\rho), \kappa)} \Gamma^{(\kappa,\pi)}, \Delta^{(\rho,\pi)}$$

where $\Delta^{(\rho,\pi)} = (\Delta^{(\kappa,\pi)})^{(\rho,\kappa)}$. Hence by an inference rule $(\kappa \in Mh_2(\vec{m}(\kappa)), j, \vec{\nu})$ we obtain (15).

Case 3. Third consider the case when the last inference is a $(\sigma \in Mh_2(\vec{\zeta}), j, \vec{\nu})$ for a $\sigma < \pi$.

$$\frac{\{(\mathcal{H}_\gamma, \Theta) \vdash_{\pi}^{a_\ell} \Gamma, \neg\delta\}_{\delta \in \Delta} \quad \{(\mathcal{H}_\gamma, \Theta \cup \{\rho\}) \vdash_{\pi}^{a_r(\rho)} \Gamma, \Delta^{(\rho,\sigma)}\}_{\rho \in H_j(\vec{\nu}, \sigma, \gamma, \Theta)}}{(\mathcal{H}_\gamma, \Theta) \vdash_{\pi}^{a} \Gamma}$$

where Δ is a finite set of $\Sigma_j(\sigma)$-sentences, and $H_j(\vec{\nu}, \sigma, \gamma, \Theta)$ is the resolvent class for $\sigma \in Mh_2(\vec{\zeta})$ with respect to j, $\vec{\nu}$, γ and Θ.

We have $\sigma < \kappa$ by (16) for $\sigma \in \mathcal{H}_\gamma[\Theta]$. Hence $\Delta \subset \Sigma_0^1(\sigma) \subset \Sigma_0(\kappa)$ and $\delta^{(\kappa,\pi)} \equiv \delta$ for any $\delta \in \Delta$. Let $H_j(\vec{\nu}, \sigma, c_1, \Theta \cup \{\kappa\})$ be the resolvent class for $\sigma \in Mh_2(\vec{\zeta})$ with respect to j, $\vec{\nu}$, c_1 and $\Theta \cup \{\kappa\}$. Then $H_j(\vec{\nu}, \sigma, c_1, \Theta \cup \{\kappa\}) \subset H_j(\vec{\nu}, \sigma, \gamma, \Theta)$.

From IH we have $(\mathcal{H}_{c_1}, \Theta \cup \{\kappa\}) \vdash_{\kappa}^{\beta(a_\ell, \kappa)} \Gamma^{(\kappa,\pi)}, \neg\delta$ for each $\delta \in \Delta$, and $(\mathcal{H}_{c_1}, \Theta \cup \{\kappa, \rho\}) \vdash_{\kappa}^{\beta(a_r(\rho), \kappa)} \Gamma^{(\kappa,\pi)}, \Delta^{(\rho,\sigma)}$ for each $\rho \in H_j(\vec{\nu}, \sigma, c_1, \Theta \cup \{\kappa\})$. We obtain (15) by the inference rule $(\sigma \in Mh_2(\vec{\zeta}), j, \vec{\nu})$ with the resolvent class $H_j(\vec{\nu}, \sigma, c_1, \Theta \cup \{\kappa\})$.

Case 4. Fourth consider the case when the last inference introduces a $\Pi_{k+1}(\pi)$-sentence $(\forall x \in L_\pi\, \theta(x)) \in \Gamma$.

$$\frac{\{(\mathcal{H}_\gamma, \Theta \cup \mathsf{k}(d)) \vdash_{\pi}^{a(d)} \Gamma, \theta(d) : d \in Tm(\pi)\}}{(\mathcal{H}_\gamma, \Theta) \vdash_{\pi}^{a} \Gamma}\ (\bigwedge)$$

Let $d \in Tm(\kappa)$ with $\mathsf{k}(d) < \kappa$ for (14).

IH yields

$$(\mathcal{H}_{c_1}, \Theta \cup \{\kappa\} \cup \mathsf{k}(d)) \vdash_{\kappa}^{\beta(a(d), \kappa)} \Gamma^{(\kappa,\pi)}, \theta(d)^{(\kappa,\pi)}$$

$(\bigwedge)$ yields (15) for $\forall x\in L_\kappa\,\theta(x)^{(\kappa,\pi)}\equiv(\forall x\in L_\pi\,\theta(x))^{(\kappa,\pi)}\in\Gamma^{(\kappa,\pi)}$.

Case 5. Fifth consider the case when the last inference introduces a $\Sigma_0(\pi)$-sentence $(\forall x\in c\,\theta(x))\in\Gamma$ for a $c\in Tm(\pi)$.

$$\frac{\{(\mathcal{H}_\gamma,\Theta\cup\mathsf{k}(d))\vdash^{a(d)}_{\pi}\Gamma,\theta(d):d\in Tm(|c|)\}}{(\mathcal{H}_\gamma,\Theta)\vdash^{a}_{\pi}\Gamma}\;(\bigwedge)$$

Then we have $|d|<|c|<\kappa$ by (16). IH yields

$$\frac{\{(\mathcal{H}_{c_1},\Theta\cup\{\kappa\}\cup\mathsf{k}(d)\vdash^{\beta(a(d),\kappa)}_{\kappa}\Gamma^{(\kappa,\pi)},\theta(d):d\in Tm(|c|)\}}{(\mathcal{H}_{c_1},\Theta\cup\{\kappa\})\vdash^{\beta(a,\kappa)}_{\kappa}\Gamma^{(\kappa,\pi)}}\;(\bigwedge)$$

Case 6. Sixth consider the case when the last inference introduces a $\Sigma_k(\pi)$-sentence $(\exists x\in L_\pi\,\theta(x))\in\Gamma$. For a $d\in Tm(\pi)$

$$\frac{(\mathcal{H}_\gamma,\Theta)\vdash^{a_0}_{\pi}\Gamma,\theta(d)}{(\mathcal{H}_\gamma,\Theta)\vdash^{a}_{\pi}\Gamma}\;(\bigvee)$$

Without loss of generality we can assume that $\mathsf{k}(d)\subset\mathsf{k}(\theta(d))$. Then we see that $|d|<\kappa$ from (16), and $d\in Tm(\kappa)$. Also $|d|<\kappa<\beta(a,\kappa)$ for (9). IH yields with $(\exists x\in L_\pi\,\theta(x))^{(\kappa,\pi)}\equiv(\exists x\in L_\kappa\,\theta(x)^{(\kappa,\pi)})\in\Gamma^{(\kappa,\pi)}$

$$\frac{(\mathcal{H}_{c_1},\Theta\cup\{\kappa\})\vdash^{\beta(a_0,\kappa)}_{\kappa}\Gamma^{(\kappa,\pi)},\theta(d)^{(\kappa,\pi)}}{(\mathcal{H}_{c_1},\Theta\cup\{\kappa\})\vdash^{\beta(a,\kappa)}_{\kappa}\Gamma^{(\kappa,\pi)}}\;(\bigvee)$$

Case 7. Seventh consider the case when the last inference is a (*cut*).

$$\frac{(\mathcal{H}_\gamma,\Theta)\vdash^{a_0}_{\pi}\Gamma,\neg C\quad(\mathcal{H}_\gamma,\Theta)\vdash^{a_0}_{\pi}C,\Gamma}{(\mathcal{H}_\gamma,\Theta)\vdash^{a}_{\pi}\Gamma}\;(cut)$$

where $a_0<a$ and $\mathrm{rk}(C)<\pi$. Then $C\in\Sigma_0(\pi)$ by Proposition 4.5.4. On the other side we have $\mathsf{k}(C)\subset\pi$ by Proposition 4.5.2. Then $\mathsf{k}(C)\subset\kappa$ by (16). Hence $C^{(\kappa,\pi)}\equiv C$ and $\mathrm{rk}(C^{(\kappa,\pi)})<\kappa$ again by Proposition 4.5.2. IH yields $(\mathcal{H}_{c_1},\Theta\cup\{\kappa\})\vdash^{\beta(a_0,\kappa)}_{\kappa}\Gamma^{(\kappa,\pi)},\neg C^{(\kappa,\pi)}$ and $(\mathcal{H}_{c_1},\Theta\cup\{\kappa\})\vdash^{\beta(a_0,\kappa)}_{\kappa}C^{(\kappa,\pi)},\Gamma^{(\kappa,\pi)}$. Hence by a (*cut*) we obtain (15).

Case 8. Eighth consider the case when the last inference is an $(\Omega\in M_2)$.

$$\frac{(\mathcal{H}_\gamma,\Theta)\vdash^{a_\ell}_{\pi}\Gamma,C\quad\{(\mathcal{H}_\gamma,\Theta\cup\{\omega\alpha\})\vdash^{a_r(\alpha)}_{\pi}\neg C^{(\alpha,\Omega)},\Gamma:\alpha<\Omega\}}{(\mathcal{H}_\gamma,\Theta)\vdash^{a}_{\pi}\Gamma}$$

where $\sup\{a_\ell+1,a_r(\alpha)+1:\alpha<\Omega\}\leq a$ and $C\in\Pi_2(\Omega)$.

We have $\omega\alpha<\kappa$ for $\alpha<\Omega$. IH with $C^{(\kappa,\pi)}\equiv C$ yields for each $\alpha<\Omega$, $(\mathcal{H}_{c_1},\Theta\cup\{\kappa,\omega\alpha\})\vdash^{\beta(a_r(\alpha),\kappa)}_{\kappa}\neg C^{(\alpha,\Omega)},\Gamma^{(\kappa,\pi)}$, and
$(\mathcal{H}_{c_1},\Theta\cup\{\kappa\})\vdash^{\beta(a_\ell,\kappa)}_{\kappa}\Gamma^{(\kappa,\pi)},C$. An $(\Omega\in M_2)$ yields (15)

All other cases are seen easily from IH. $\square$

Lemma 5.3 *Let $\lambda \leq \pi$ be regular ordinal terms, and $\Gamma \subset \Sigma_1(\lambda)$.*

Suppose for an ordinal term a

$$(\mathcal{H}_\gamma, \Theta) \vdash^a_\pi \Gamma$$

where $\{\gamma, \lambda, \pi\} \subset \mathcal{H}_\gamma[\Theta]$.

Assume

$$\forall \rho \in [\lambda, \pi] \forall d > 0 [\Theta \subset \psi_\rho(\gamma \# d)] \tag{26}$$

Let $\hat{a} = \gamma \# \omega^{\pi+a+1}$ and $\beta = \psi_\lambda(\hat{a})$. Then the following holds

$$(\mathcal{H}_{\hat{a}+1}, \Theta) \vdash^\beta_\beta \Gamma \tag{27}$$

Proof by main induction on π with subsidiary induction on $a > 0$.

First we show, cf. (13), that

$$\lambda \leq \tau = \psi^{\vec{\zeta}}_\sigma(a_0) \in \mathcal{H}_\gamma[\Theta] \Rightarrow \forall \langle b, \rho, c \rangle \in K^2(\vec{\zeta})(b \in \mathcal{H}_\gamma[\Theta] \,\&\, c \leq \gamma) \tag{28}$$

Let $\lambda \leq \tau = \psi^{\vec{\zeta}}_\sigma(a_0) \in \mathcal{H}_\gamma[\Theta]$. Then by (26) we have $\Theta \subset \lambda \subset \tau$, and hence for any $\langle b, \rho, c \rangle \in K^2(\vec{\zeta})$ we have $b \in \mathcal{H}_\gamma[\Theta]$ and $c \leq a_0 < \gamma$.

We see that $\Theta \subset \beta = \psi_\lambda(\hat{a})$ from (26). Hence

$$a_0 \in \mathcal{H}_\gamma[\Theta] \cap a \Rightarrow \psi_\lambda(\widehat{a_0}) < \psi_\lambda(\hat{a})$$

By the assumption (26), (13) and (8) we have

$$\forall \rho \in [\lambda, \pi] \forall \langle b, \rho, c \rangle \in K^2(\vec{\xi}) \forall d > 0 [\mathsf{k}(\Gamma) \cup \{\gamma, \lambda, a, \pi, b\} \subset \mathcal{H}_\gamma(\psi_\rho(\gamma \# d))] \tag{29}$$

Let $\vec{\xi}$ and k denote a sequence of ordinals and a number defined as follows. If $\pi = \mathbb{K}$, then let $\vec{\xi} = \vec{0}$ with $lh(\vec{\xi}) = N - 1$ and $k = N - 1$. If $\pi < \mathbb{K}$, then let $\vec{\xi} = \vec{m}(\pi)$. In each case let $k = \max(\{1\} \cup \{k \leq N - 2 : \xi_{k+1} > 0\})$ for $\vec{\xi} = (\xi_2, \ldots, \xi_{N-1}) = (\xi_2, \ldots, \xi_{k+1}) * \vec{0}$.

Since $\pi \in \mathcal{H}_\gamma[\Theta]$, we have (13) for $\vec{\xi}, \gamma, \Theta$ by (28).

Case 1. First consider the case when $k \geq 2$.

Let $\vec{\zeta}(a) := \vec{\zeta^c}(a) = (\zeta_2(a), \ldots, \zeta_k(a)) * \vec{0}$ be the strongly irreducible sequence defined as in Lemma 5.1 for $c = \gamma + a$: $\vec{\zeta}(a) = \vec{0} * (\langle \gamma + a, \mathbb{K}, \gamma + a \rangle)$ when $\pi = \mathbb{K}$, otherwise $\zeta_k(a) = \xi_k + \Lambda^{\xi_{k+1}} \langle \gamma + a, \pi, \gamma + a \rangle$ and $\zeta_i(a) = \xi_i$ for $i < k$. Also let $\gamma(a, b) = \gamma \# a \# b$ and $\beta(a, b) = \psi_\pi \gamma(a, b)$.

We have $\forall \langle b, \rho, c \rangle \in K^2(\vec{\xi})(b \leq c \leq \gamma)$ by Proposition 3.4. This yields $\forall \langle b, \rho, c \rangle \in K^2(\vec{\zeta}(a))(c \leq \gamma(a, 0) = \gamma \# a)$. Let $\kappa := \psi^{\vec{\zeta}(a)}_\pi(\gamma(a, 0))$. By the assumption (26) and $a > 0$ we have $\Theta \subset \psi_\pi(\gamma \# a)$. On the other hand we have $\psi_\pi(\gamma \# a) \leq \kappa$, and hence (14), $\Theta \subset \kappa$. From this we see that $\forall \langle b, \rho, c \rangle \in K^2(\vec{\zeta}(a))(b \in \mathcal{H}_{\gamma(a,0)}(\kappa))$ and $\{\pi, \gamma(a, 0)\} \subset \mathcal{H}_{\gamma(a,0)}(\kappa)$. Therefore $\kappa \in OT$ by Definition 3.3.15 such that $\kappa < \pi$ and $\mathcal{H}_\gamma(\kappa) \cap \pi \subset \kappa$.

By Lemma 5.1 we have for $c_1 = \gamma(a, \kappa) + 1 \geq \gamma + a = c$

$$(\mathcal{H}_{\gamma(a,\kappa)+1}, \Theta \cup \{\kappa\}) \vdash^{\beta(a,\kappa)}_\kappa \Gamma^{(\kappa,\pi)}$$

and by $\kappa\in\mathcal{H}_{\gamma(a,0)+1}[\Theta]$ and Lemma 4.7.2 we obtain

$$(\mathcal{H}_{\gamma(a,\kappa)+1},\Theta)\vdash^{\beta(a,\kappa)}_{\kappa}\Gamma^{(\kappa,\pi)} \tag{30}$$

If $\lambda=\pi$, then $\Gamma^{(\kappa,\pi)}\subset\Sigma_1(\kappa)\subset\Sigma_0(\lambda)$. We have $\Theta\subset\psi_\pi(\hat{a})=\beta$, and $\kappa\in\mathcal{H}_{\hat{a}}(\beta)$. Hence $\{\gamma,\pi,a,\kappa\}\subset\mathcal{H}_{\hat{a}}(\beta)$, and $\gamma(a,\kappa)=\gamma\#a\#\kappa<\gamma\#\omega^{\pi+a+1}=\hat{a}$. Therefore $\kappa<\beta(a,\kappa)\leq\psi_\pi(\hat{a})=\beta$. We obtain (27) by Persistency lemma 4.11.

Next consider the case when $\lambda<\pi$. Then $\lambda<\kappa$ and $\Gamma^{(\kappa,\pi)}=\Gamma$. We have for (26), $\forall d\forall\rho\in[\lambda,\kappa](\Theta\subset\psi_\rho(\gamma(a,\kappa)+1\#d))$. By MIH on (30) we have for $\beta_0=\psi_\lambda(b_0)$ with $b_0=(\gamma(a,\kappa)+1)\#\omega^{\kappa+\beta(a,\kappa)+1}$

$$(\mathcal{H}_{b_0+1},\Theta)\vdash^{\beta_0}_{\beta_0}\Gamma$$

We have $b_0=\gamma\#a\#\kappa\#1\#\omega^{\beta(a,\kappa)+1}<\gamma\#\omega^{\pi+a+1}=\hat{a}$ by $\beta(a,\kappa)<\pi$. This yields $\psi_\lambda(b_0)=\beta_0<\beta=\psi_\lambda(\hat{a})$ by $\Theta\subset\beta$ and $\{\gamma,\kappa,\pi,a\}\subset\mathcal{H}_{\hat{a}}(\beta)$. Hence (27) follows.

In what follows suppose $k=1$.

Case 2. Consider the case when the last inference rule is a $(\pi\in Mh_2(\vec{\xi}),2,\vec{\nu})$.

We have $a_\ell\in\mathcal{H}_\gamma[\Theta]\cap a$, and $a_r(\rho)\in\mathcal{H}_\gamma[\Theta\cup\{\rho\}]\cap a$. Δ is a finite set of $\Sigma_2(\pi)$-sentences.

$$\frac{\{(\mathcal{H}_\gamma,\Theta)\vdash^{a_\ell}_{\pi}\Gamma,\neg\delta\}_{\delta\in\Delta}\quad\{(\mathcal{H}_\gamma,\Theta\cup\{\rho\})\vdash^{a_r(\rho)}_{\pi}\Gamma,\Delta^{(\rho,\pi)}\}_{\rho\in H_2(\vec{\nu},\pi,\gamma,\Theta)}}{(\mathcal{H}_\gamma,\Theta)\vdash^{a}_{\pi}\Gamma}$$

$H_2(\vec{\nu},\pi,\gamma,\Theta)$ is the resolvent class for $\pi\in Mh_2(\vec{\xi})$ with respect to 2, $\vec{\nu}=(\nu_2,\ldots,\nu_{N-1})$, γ and Θ:

$$H_2(\vec{\nu},\pi,\gamma,\Theta)=\{\rho\in Mh_2(\vec{\nu})\cap\pi:\mathcal{H}_\gamma(\rho)\cap\pi\subset\rho\,\&\,\Theta\cap\pi\subset\rho\}$$

where $\vec{\nu}$ is a strongly irreducible sequence such that $\vec{\nu}<_{Ksl}\xi_2$ with $\vec{m}(\pi)=(\xi_2)*\vec{0}$, $\forall\langle b,\rho,c\rangle\in K^2(\vec{\xi})\cup K^2(\vec{\nu})(\mathcal{H}_\gamma[\Theta\cap\pi]\ni b\leq c\leq\gamma)$.

Let for $\widehat{a_\ell}=\gamma\#\omega^{\pi+a_\ell+1}$, $\rho=\psi^{\vec{\nu}}_{\pi}(\widehat{a_\ell}\#\pi)$. By the assumption (26) we have $\Theta\subset\psi_\pi(\widehat{a_\ell})\subset\rho$. Also $\forall\langle b,\sigma,c\rangle\in K^2(\vec{\nu})(b\in\mathcal{H}_c(\Theta)\subset\mathcal{H}_c(\rho)\,\&\,c\leq\widehat{a_\ell})$ by (11), and $\{\pi,\widehat{a_\ell}\}\subset\mathcal{H}_\gamma(\rho)$. Furthermore let $\langle b,\sigma,c\rangle\in K^2(\vec{\nu})$. Then $b\in\mathcal{H}_c(\Theta)\subset\mathcal{H}_c(\pi)$. Hence $b\in\mathcal{H}_c(\pi)$, i.e., $K_\pi(b)<c$. The condition (7) in Definition 3.3.16c is fulfilled. Therefore $\rho\in OT$ by Definition 3.3.16. We have shown $\rho\in H_2(\vec{\nu},\pi,\gamma,\Theta)$.

By Inversion lemma 4.9 we obtain for each $\delta\equiv(\exists x\in L_\pi\delta_1(x))\in\Delta$ and each $d\in Tm(\rho)$ with $|d|=\max(\{0\}\cup\mathsf{k}(d))$

$$(\mathcal{H}_{\gamma\#|d|},\Theta\cup\mathsf{k}(d))\vdash^{a_\ell}_{\pi}\Gamma,\neg\delta_1(d)$$

We have $\{\pi,\gamma,|d|\}\subset\mathcal{H}_{\gamma\#|d|}(\pi)$ by $|d|<\rho<\pi$, and this yields $|d|\in\mathcal{H}_{\gamma\#|d|}(\psi_\pi(\gamma\#|d|))\cap\pi\subset\psi_\pi(\gamma\#|d|)$. Hence $|d|<\psi_\pi(\gamma\#|d|)$, and $\forall e>0(\Theta\cup$

$\mathsf{k}(d) \subset \psi_\pi(\gamma\#|d|\#e))$, i.e., (26) holds for $\lambda = \pi$ and $\gamma\#|d|$. Let $\beta_d = \psi_\pi(\widehat{a_d})$ for $\widehat{a_d} = \gamma\#|d|\#\omega^{\pi+a_\ell+1} = \widehat{a_\ell}\#|d|$. SIH yields

$$(\mathcal{H}_{\widehat{a_d}+1}, \Theta \cup \mathsf{k}(d)) \vdash^{\beta_d}_{\beta_d} \Gamma, \neg\delta_1(d)$$

By Boundedness lemma 4.10 we have for $\widehat{a_\pi} = \gamma\#\pi\#\omega^{\pi+a_\ell+1} = \widehat{a_\ell}\#\pi$

$$(\mathcal{H}_{\widehat{a_\pi}+1}, \Theta \cup \mathsf{k}(d)) \vdash^{\beta_d}_{\beta_d} \Gamma, \neg\delta_1^{(\beta_d,\pi)}(d)$$

By persistency we obtain for $\beta_d < \rho \in \mathcal{H}_\gamma[\Theta]$

$$(\mathcal{H}_{\widehat{a_\pi}+1}, \Theta \cup \mathsf{k}(d)) \vdash^{\beta_d}_{\rho} \Gamma, \neg\delta_1^{(\rho,\pi)}(d)$$

Since $d \in Tm(\rho)$ is arbitrary, $(\bigwedge)$ yields

$$(\mathcal{H}_{\widehat{a_\pi}+1}, \Theta) \vdash^{\rho}_{\rho} \Gamma, \neg\delta^{(\rho,\pi)} \tag{31}$$

Now pick the ρ-th branch from the right upper sequents

$$(\mathcal{H}_{\widehat{a_\pi}+1}, \Theta \cup \{\rho\} \vdash^{a_r(\rho)}_{\pi} \Gamma, \Delta^{(\rho,\pi)}$$

By $\rho \in \mathcal{H}_{\widehat{a_\pi}+1}[\Theta]$ and Lemma 4.7.2 we obtain

$$(\mathcal{H}_{\widehat{a_\pi}+1}, \Theta) \vdash^{a_r(\rho)}_{\pi} \Gamma, \Delta^{(\rho,\pi)} \tag{32}$$

Case 2.1. First consider the case $\lambda = \pi$. Then $\Delta^{(\rho,\pi)} \subset \Sigma_0(\lambda)$. Let $\beta_\rho = \psi_\pi(b_\rho)$ with $b_\rho = \widehat{a_\pi}\#1\#\omega^{\pi+a_r(\rho)+1} = \gamma\#\omega^{\pi+a_\ell+1}\#\omega^{\pi+a_r(\rho)+1}\#\pi\#1$. Then $\beta_\rho > \rho$ and $\forall d[\Theta \cup \{\rho\} \subset \psi_\pi(\widehat{a_\pi}+1\#d)]$. SIH yields for (32)

$$(\mathcal{H}_{b_\rho+1}, \Theta) \vdash^{\beta_\rho}_{\beta_\rho} \Gamma, \Delta^{(\rho,\pi)} \tag{33}$$

Several (cut)'s yield with (33), (31) and for $\beta_\rho \geq \rho$, $\widehat{a_\pi} < b_\rho < \hat{a}$ and some $p < \omega$

$$(\mathcal{H}_{\hat{a}+1}, \Theta) \vdash^{\beta_\rho+p}_{\beta_\rho} \Gamma$$

where $\beta_\rho < \beta = \psi_\pi(\hat{a})$ by $b_\rho < \hat{a}$. (27) follows.

Case 2.2. Next consider the case when $\lambda < \pi$. Then $\lambda < \rho$ and $\Delta^{(\rho,\pi)} \subset \Sigma_1(\rho^+)$. For (26) we have $\rho < \psi_\pi(\widehat{a_\ell}+1)$ and $\rho < \psi_\sigma(\widehat{a_\ell}+1+d)$ for any $d > 0$ and any σ with $\rho^+ \leq \sigma \leq \pi$ by $\{b, \pi, \widehat{a_\ell}\} \subset \mathcal{H}_\gamma[\Theta] \subset \mathcal{H}_\gamma(\psi_\sigma(\widehat{a_\ell}+1))$ for any $\langle b, \sigma, c\rangle \in K^2(\vec{\nu})$. SIH yields for $\beta_{\rho^+} = \psi_{\rho^+}(b_\rho) > \rho$ and (32)

$$(\mathcal{H}_{b_\rho+1}, \Theta \cup \{\rho\}) \vdash^{\beta_{\rho^+}}_{\beta_{\rho^+}} \Gamma, \Delta^{(\rho,\pi)}$$

and by Lemma 4.7.2

$$(\mathcal{H}_{b_\rho+1}, \Theta) \vdash^{\beta_{\rho^+}}_{\beta_{\rho^+}} \Gamma, \Delta^{(\rho,\pi)} \tag{34}$$

Several (cut)'s yield with (31), (34) and for $\beta_{\rho^+} > \rho$ and $b_0 = \gamma\#(\omega^{\pi+a_\ell+1}\cdot 2)\#\omega^{\pi+a_r(\rho)+1}\#1 \geq \max\{b_\ell, b_\rho\}$

$$(\mathcal{H}_{b_0+1}, \Theta) \vdash^{\beta_{\rho^+}+p}_{\beta_{\rho^+}} \Gamma$$

Predicative cut-elimination lemma 4.12 yields for $\beta_1 = \varphi(\beta_{\rho^+})(\beta_{\rho^+}+p) < \rho^+$

$$(\mathcal{H}_{b_0+1}, \Theta) \vdash^{\beta_1}_{\rho} \Gamma \tag{35}$$

We have $\lambda < \rho \in \mathcal{H}_{b_0+1}[\Theta]$ by $\gamma < \widehat{a_\ell} < b_0$. As to (26), $\forall d \forall \sigma \in [\lambda, \rho][\Theta \subset \psi_\sigma(b_0+1+d)]$, which holds by $b_0 > \gamma$ and $\rho < \pi$.

Hence MIH with (35) yields for $c = b_0\#1\#\omega^{\rho+\beta_1+1}$

$$(\mathcal{H}_{c+1}, \Theta) \vdash^{\psi_\lambda c}_{\psi_\lambda c} \Gamma$$

We have $c = b_0\#\omega^{\rho+\beta_1+1}\#1 = \gamma\#(\omega^{\pi+a_\ell+1}\cdot 2)\#\omega^{\pi+a_r(\rho)+1}\#\omega^{\rho+\beta_1+1}\#2 < \gamma\#\omega^{\pi+a+1} = \hat{a}$ since $a_\ell, a_r(\rho) < a$ and $\rho, \beta_1 < \rho^+ < \pi$. Hence $\psi_\lambda c < \psi_\lambda(\hat{a}) = \beta$, and (27) follows.

Case 3. Third consider the case when the last inference introduces a $\Sigma_1(\lambda)$-sentence $(\forall x \in c\, \theta(x)) \in \Gamma$ for $c \in Tm(\lambda)$.

$$\frac{\{(\mathcal{H}_\gamma, \Theta \cup \mathsf{k}(d)) \vdash^{a(d)}_{\pi} \Gamma, \theta(d) : d \in Tm(|c|)\}}{(\mathcal{H}_\gamma, \Theta) \vdash^{a}_{\pi} \Gamma}\ (\bigwedge)$$

Then we see from (29) that $|d| < |c| \in \mathcal{H}_\gamma(\psi_\rho(\gamma\#e)) \cap \rho \subset \psi_\rho(\gamma\#e)$ for any $\rho \in [\lambda, \pi]$ and any $e > 0$. Hence $|d| \in \psi_\rho(\gamma\#e)$. (26) is enjoyed for $\Theta \cup \mathsf{k}(d)$. SIH yields for $\beta_d = \psi_\lambda(\widehat{a(d)})$

$$(\mathcal{H}_{\hat{a}+1}, \Theta \cup \mathsf{k}(d)) \vdash^{\beta_d}_{\beta_d} \Gamma, \theta(d)$$

$(\bigwedge)$ yields (27) for $\beta = \psi_\lambda(\hat{a}) > \beta_d$.

Case 4. Fourth consider the case when the last inference introduces a $\Sigma_1(\lambda)$-sentence $(\exists x \in L_\lambda\, \theta(x)) \in \Gamma$. For a $d \in Tm(\lambda)$

$$\frac{(\mathcal{H}_\gamma, \Theta) \vdash^{a_0}_{\pi} \Gamma, \theta(d)}{(\mathcal{H}_\gamma, \Theta) \vdash^{a}_{\pi} \Gamma}\ (\bigvee)$$

Without loss of generality we can assume that $\mathsf{k}(d) \subset \mathsf{k}(\theta(d))$. Then we see from (29) that $|d| \in \mathcal{H}_\gamma(\psi_\lambda(\gamma+1)) \cap \lambda \subset \psi_\lambda(\gamma+1) < \beta$. Thus (9) is enjoyed in the following inference rule $(\bigvee)$. SIH yields for $\beta = \psi_\lambda(\hat{a}) > \psi_\lambda(\widehat{a_0}) = \beta_0$

$$\frac{(\mathcal{H}_{\hat{a}+1}, \Theta) \vdash^{\beta_0}_{\beta_0} \Gamma, \theta(d)}{(\mathcal{H}_{\hat{a}+1}, \Theta) \vdash^{\beta}_{\beta} \Gamma}\ (\bigvee)$$

Case 5. Fifth consider the case when the last inference is a $(\tau \in Mh_2(\vec{\iota}), j, \vec{\nu})$ for a $\tau \in \mathcal{H}_\gamma[\Theta] \cap \pi$ and $\vec{\iota} = (\iota_2, \ldots, \iota_{N-1})$:

$$\frac{\{(\mathcal{H}_\gamma, \Theta) \vdash^{a_\ell}_\pi \Gamma, \neg\delta\}_{\delta\in\Delta} \quad \{(\mathcal{H}_\gamma, \Theta \cup \{\rho\}) \vdash^{a_r(\rho)}_\pi \Gamma, \Delta^{(\rho,\tau)}\}_{\rho\in H_j(\vec{\nu},\tau,\gamma,\Theta)}}{(\mathcal{H}_\gamma, \Theta,) \vdash^a_\pi \Gamma}$$

where Δ is a finite set of $\Sigma_j(\tau)$-sentences, and $H_j(\vec{\nu}, \tau, \gamma, \Theta) = \{\rho \in Mh_2(\vec{\nu}) \cap \tau : \mathcal{H}_\gamma(\rho) \cap \tau \subset \rho \,\&\, \Theta \cap \tau \subset \rho\}$ is the resolvent class for $\tau \in Mh_2(\vec{\iota})$ with respect to j, $\vec{\nu}$, γ and Θ. By (29), for any $\rho \in H_j(\vec{\nu}, \tau, \gamma, \Theta)$ we have

$$\forall e > 0 \forall \kappa[\max\{\tau+1, \lambda\} \leq \kappa \leq \pi \Rightarrow \rho < \tau \in \mathcal{H}_\gamma(\psi_\kappa(\gamma\#e)) \cap \kappa \subset \psi_\kappa(\gamma\#e)] \quad (36)$$

Case 5.1. First consider the case when $\tau < \lambda$. Then $\rho < \psi_\kappa(\gamma\#e)$ for any $\kappa \in [\lambda, \pi]$ and $e > 0$. From SIH with (36) we obtain the lemma by an inference rule $(\tau \in Mh_2(\vec{\iota}), j, \vec{\nu})$ with the resolvent class $H_j(\vec{\nu}, \tau, \gamma, \Theta)$ for $\beta_\ell = \psi_\lambda(\widehat{a_\ell}), \beta_r(\rho) = \psi_\lambda(\widehat{a_r(\rho)}), \tau < \beta = \psi_\lambda(\hat{a})$.

$$\frac{\{(\mathcal{H}_{\hat{a}+1}, \Theta) \vdash^{\beta_\ell}_{\beta_\ell} \Gamma, \neg\delta\}_{\delta\in\Delta} \quad \{(\mathcal{H}_{\hat{a}+1}, \Theta \cup \{\rho\}) \vdash^{\beta_r(\rho)}_{\beta_r(\rho)} \Gamma, \Delta^{(\rho,\tau)}\}_{\rho\in H_j(\vec{\nu},\tau,\gamma,\Theta)}}{(\mathcal{H}_{\hat{a}+1}, \Theta) \vdash^\beta_\beta \Gamma}$$

Case 5.2. Second consider the case when $\lambda \leq \tau$. Then $\Delta \cup \Delta^{(\rho,\tau)} \subset \Sigma_1(\tau^+)$, and $\rho < \psi_\kappa(\gamma\#e)$ for $\tau < \kappa \leq \pi$ and $e > 0$ by (36). SIH yields for $\beta_2 = \psi_{\tau^+}\widehat{a_\ell}$ and $\beta_\rho = \psi_{\tau^+}\widehat{a_r(\rho)}$

$$\{(\mathcal{H}_{\widehat{a_\ell}+1}, \Theta) \vdash^{\beta_2}_{\beta_2} \Gamma, \neg\delta\}_{\delta\in\Delta}$$

We see similarly from SIH that

$$\{(\mathcal{H}_{\widehat{a_r(\rho)}+1}, \Theta \cup \{\rho\}) \vdash^{\beta_\rho}_{\beta_\rho} \Gamma, \Delta^{(\rho,\tau)}\}_{\rho\in H_j(\vec{\nu},\tau,\gamma,\Theta)}$$

Predicative cut-elimination lemma 4.12 yields for $\delta_2 = \varphi(\beta_2)(\beta_2)$ and $\delta_\rho = \varphi(\beta_\rho)(\beta_\rho)$

$$\{(\mathcal{H}_{\widehat{a_\ell}+1}, \Theta) \vdash^{\delta_2}_\tau \Gamma, \neg\delta\}_{\delta\in\Delta}$$

and

$$\{(\mathcal{H}_{\widehat{a_r(\rho)}+1}, \Theta \cup \{\rho\}) \vdash^{\delta_\rho}_\tau \Gamma, \Delta^{(\rho,\tau)}\}_{\rho\in H_j(\vec{\nu},\tau,\gamma,\Theta)}$$

From these with the inference rule $(\tau \in Mh_2(\vec{\iota}), j, \vec{\nu})$ we obtain

$$(\mathcal{H}_{\widehat{a_0}+1}, \Theta) \vdash^{\delta_0+1}_\tau \Gamma \quad (37)$$

where $\sup\{\delta_2, \delta_\rho : \rho \in H_j(\vec{\nu}, \tau, \widehat{a_0} + 1, \Theta)\} \leq \delta_0 := \varphi(\beta_0)(\beta_0) \in \mathcal{H}_{\widehat{a_0}+1}[\Theta]$ with $\sup\{\beta_2, \beta_\rho : \rho \in H_j(\vec{\nu}, \tau, \gamma, \Theta)\} \leq \beta_0 := \psi_{\tau^+}\widehat{a_0}$, and $\sup\{a_\ell, a_r(\rho) : \rho \in H_j(\vec{\nu}, \tau, \gamma, \Theta)\} \leq a_0 \in \mathcal{H}_\gamma[\Theta] \cap a$, cf. (12).

As to (26), $\{\widehat{a_0}, \tau\} \subset \mathcal{H}_{\widehat{a_0}+1}[\Theta]$, and $\forall e \forall \rho \in [\lambda, \tau][\Theta \subset \psi_\rho(\widehat{a_0} + 1\#e)]$.

MIH with (37) and (28) yields for $\delta = \psi_\lambda(\widehat{a_0}\#\omega^{\tau+\delta_0+2})$

$$(\mathcal{H}_{\hat{a}+1}, \Theta) \vdash^\delta_\delta \Gamma$$

We have $\delta = \psi_\lambda(\widehat{a_0}\#\omega^{\tau+\delta_0+2}) < \psi_\lambda(\hat{a}) = \beta$ by $\widehat{a_0} < \hat{a}$ and $\tau, \delta_0 < \tau^+ < \pi$ and $\tau \in \mathcal{H}_\gamma[\Theta]$. (27) follows.

Case 6. Sixth consider the case when the last inference is a (cut). For an $a_0 < a$ and a C with $\mathrm{rk}(C) < \pi$.

$$\frac{(\mathcal{H}_\gamma, \Theta) \vdash^{a_0}_{\pi} \Gamma, \neg C \quad (\mathcal{H}_\gamma, \Theta) \vdash^{a_0}_{\pi} C, \Gamma}{(\mathcal{H}_\gamma, \Theta) \vdash^{a}_{\pi} \Gamma}\ (cut)$$

Case 6.1. First consider the case when $\mathrm{rk}(C) < \lambda$. Then $C \in \Sigma_0(\lambda)$. SIH yields the lemma.

Case 6.2. Second consider the case when $\lambda \leq \mathrm{rk}(C) < \pi$. Let $\rho^+ = (\mathrm{rk}(C))^+ = \min\{\kappa \in Reg : \mathrm{rk}(C) < \kappa\}$. Then $C \in \Sigma_0(\rho^+)$ and $\lambda \leq \rho \in \mathcal{H}_\gamma[\Theta] \cap \pi$. SIH yields for $\beta_0 = \psi_{\rho^+}\widehat{a_0} \in \mathcal{H}_{\widehat{a_0}+1}[\Theta]$

$$(\mathcal{H}_{\widehat{a_0}+1}, \Theta) \vdash^{\beta_0}_{\beta_0} \Gamma, \neg C$$

and

$$(\mathcal{H}_{\widehat{a_0}+1}, \Theta) \vdash^{\beta_0}_{\beta_0} C, \Gamma$$

By a (cut) we obtain for $\beta_1 = \max\{\beta_0, \mathrm{rk}(C)\} + 1$ with $\rho < \beta_1 < \rho^+$

$$(\mathcal{H}_{\widehat{a_0}+1}, \Theta) \vdash^{\beta_1}_{\beta_1} \Gamma$$

Predicative cut-elimination lemma 4.12 yields for $\delta_1 = \varphi(\beta_1)(\beta_1)$

$$(\mathcal{H}_{\widehat{a_0}+1}, \Theta) \vdash^{\delta_1}_{\rho} \Gamma,$$

where we have $\widehat{a_0} \in \mathcal{H}_{\widehat{a_0}+1}[\Theta]$, and $\forall e \forall \tau \in [\lambda, \rho][\Theta \subset \psi_\tau(\widehat{a_0}\#e)]$.

Hence MIH with $\rho \in \mathcal{H}_{\widehat{a_0}+1}[\Theta]$ and (28) yields for $b = \widehat{a_0}\#1\#\omega^{\rho+\delta_1+1}$

$$(\mathcal{H}_{b+1}, \Theta) \vdash^{\psi_\lambda(b)}_{\psi_\lambda(b)} \Gamma$$

We have $b < \hat{a}$ and $\psi_\lambda(b) < \psi_\lambda(\hat{a}) = \beta$, and (27) follows.

Case 7. Seventh consider the case when the last inference is an $(\Omega \in M_2)$.

$$\frac{(\mathcal{H}_\gamma, \Theta) \vdash^{a_\ell}_{\pi} \Gamma, C \quad \{(\mathcal{H}_\gamma, \Theta \cup \{\alpha\}) \vdash^{a_r(\alpha)}_{\pi} \neg C^{(\alpha,\Omega)}, \Gamma : \alpha < \Omega\}}{(\mathcal{H}_\gamma, \Theta) \vdash^{a}_{\pi} \Gamma}$$

where $C \in \Pi_2(\Omega)$.

The case $\lambda > \Omega$ is seen as in **Case 5.1**. The case $\lambda = \Omega$ is seen as in **Case 5.2**.

$\square$

Let us conclude Theorem 1.1. Let $\Omega = \Omega_1$.

Proof of Theorem 1.1. Let $\mathsf{KP\Pi}_N \vdash \theta$. By Embedding lemma 4.14 pick an m so that
$$(\mathcal{H}_0, \emptyset) \vdash^{\mathbb{K}\cdot 2+m}_{\mathbb{K}+m} \theta.$$
Predicative cut-elimination lemma 4.12 yields for $\omega_m(\mathbb{K}\cdot 2+m) < \omega_{m+1}(\mathbb{K}+1)$,
$$(\mathcal{H}_0, \emptyset) \vdash^{\omega_{m+1}(\mathbb{K}+1)}_{\mathbb{K}} \theta.$$
Lemma 5.3 yields for $a = \omega^{\mathbb{K}+\omega_{m+1}(\mathbb{K}+1)+1}$ and $\beta = \psi_\Omega(a)$
$$(\mathcal{H}_{a+1}, \emptyset) \vdash^{\beta}_{\beta} \theta$$
Predicative cut-elimination lemma 4.12 yields
$$(\mathcal{H}_{a+1}, \emptyset) \vdash^{\varphi(\beta)(\beta)}_{0} \theta$$
We have $\varphi(\beta)(\beta) < \alpha := \psi_\Omega(\omega_n(\mathbb{K}+1))$ for $n = m+3$, and hence
$$(\mathcal{H}_{\omega_n(\mathbb{K}+1)}, \emptyset) \vdash^{\alpha}_{0} \theta$$
Boundedness lemma 4.10 yields
$$(\mathcal{H}_{\omega_n(\mathbb{K}+1)}, \emptyset) \vdash^{\alpha}_{0} \theta^{(\alpha,\Omega)}$$
Since each inference rule other than reflection rules $(\pi \in Mh_2(\vec{\xi}), k, \vec{\nu})$, is sound, we see by induction up to $\alpha = \psi_\Omega(\omega_n(\mathbb{K}+1))$ that $L_\alpha \models \theta$. □

6 Distinguished sets

In what follows we show the Theorem 1.2. Henceforth except subsection 7.1 we consider ordinal terms *without* decorations.

Let us begin with some elementary facts on notation system OT.

Proposition 6.1 *1.* $\alpha \leq \beta \Rightarrow K_\alpha(\gamma) \supset K_\beta(\gamma)$.

2. Let $\beta = \psi^{\vec{\nu}}_{\pi}(b)$ *with* $\pi = \psi^{\vec{\xi}}_{\kappa}(a)$. *Then* $a < b$.

3. If $\kappa < \psi^{\vec{\nu}}_{\pi}(b) < \kappa^+$, *then* $\pi = \kappa^+$*(, and* $\vec{\nu} = \vec{0}$*).*

Proof. 6.1.1 is seen by induction on $\ell\gamma$.

6.1.2. Let $\beta = \psi^{\vec{\nu}}_{\pi}(b)$ with $\pi = \psi^{\vec{\xi}}_{\kappa}(a)$. Then $K_\beta(\{\pi, b\} \cup K^2(\vec{\nu})) < b$. On the other hand we have $\beta < \pi$. Hence $a \in K_\beta(\pi) < b$.

6.1.3. Let $\kappa < \psi^{\vec{\nu}}_{\pi}(b) < \kappa^+$. If $\vec{\nu} \neq \vec{0}$, then $\kappa^+ < \psi^{\vec{\nu}}_{\pi}(b)$. Hence $\vec{\nu} = \vec{0}$. Let $\kappa = \Omega_a \geq a$ with $\kappa^+ = \Omega_{a+1}$. Then $a \in \mathcal{H}_b(\psi_\pi(b))$, and $\Omega_{a+1} \in \mathcal{H}_b(\psi_\pi(b))$. If $\kappa^+ = \Omega_{a+1} < \pi$, then $\kappa^+ < \psi_\pi(b)$. Hence $\kappa < \pi \leq \kappa^+$, and $\pi = \kappa^+$. □

Proposition 6.2 $\alpha \leq \beta < \kappa^+ \Rightarrow K_\kappa\alpha \leq K_\kappa\beta$.

Proof by induction on $\ell\alpha + \ell\beta$.

We can assume $\kappa \leq \alpha$, $\beta = \psi_{\kappa^+}(b)$ and either $\alpha = \kappa$ or $\alpha = \psi_{\kappa^+}(a)$ by IH and Proposition 6.1.3. We have $K_\kappa(\{\kappa, \kappa^+\}) = \emptyset$. Let $\alpha = \psi_{\kappa^+}(a)$. Then $K_\kappa\alpha = \{a\} \cup K_\kappa(\{\kappa^+, a\})$, and by Proposition 2.17 we have $a < b$ and $a \in \mathcal{H}_b(\beta)$, i.e., $K_\beta(a) < b$ (or $K_\alpha(a) < a$ by $\alpha \in OT$). Let $c \in K_\kappa a \setminus K_\beta a$. This means that there exists a subterm $\psi_{\kappa^+}(c)$ of a such that $\kappa \leq \psi_{\kappa^+}(c) < \beta$. By IH we have $\{c\} \cup K_\kappa c = K_\kappa(\psi_{\kappa^+}(c)) \leq K_\kappa\beta$. □

6.1 Coefficients

In this subsection we introduce coefficient sets $\mathcal{E}(\alpha), G_\kappa(\alpha), F_\delta(\alpha), k_\delta(\alpha)$ of $\alpha \in OT$, each of which is a finite set of subterms of α. These are utilized in our wellfoundedness proof in section 6. Roughly $\mathcal{E}(\alpha)$ is the set of subterms of the form $\psi_\pi^{\vec{\nu}}(a)$, and $F_\delta(\alpha)$ $[k_\delta(\alpha)]$ the set of subterms$< \delta$ [subterms$\geq \delta$], resp. $G_\kappa(\alpha)$ is an analogue of sets $K_\kappa\alpha$ in [1].

Let $pd(\psi_\pi^{\vec{\nu}}(a)) = \pi$ (even if $\vec{\nu} = \vec{0}$). Moreover for n, $pd^{(n)}(\alpha)$ is defined recursively by $pd^{(0)}(\alpha) = \alpha$ and $pd^{(n+1)}(\alpha) \simeq pd(pd^{(n)}(\alpha))$.

For terms $\pi, \kappa \in OT$, $\pi \prec \kappa$ denotes the transitive closure of the relation $\{(\pi, \kappa) : \exists\vec{\xi}\exists b[\pi = \psi_\kappa^{\vec{\xi}}(b)]\}$, and its reflexive closure $\pi \preceq \kappa :\Leftrightarrow \pi \prec \kappa \lor \pi = \kappa$.

Definition 6.3 *For terms $\alpha, \kappa, \delta \in OT$, finite sets $\mathcal{E}(\alpha), G_\kappa(\alpha), F_\delta(\alpha), k_\delta(\alpha) \subset OT$ are defined recursively as follows.*

1. $\mathcal{E}(\alpha) = \emptyset$ for $\alpha \in \{0, \mathbb{K}\}$. $\mathcal{E}(\alpha_m + \cdots + \alpha_0) = \bigcup_{i \leq m} \mathcal{E}(\alpha_i)$. $\mathcal{E}(\varphi\beta\gamma) = \mathcal{E}(\beta) \cup \mathcal{E}(\gamma)$. $\mathcal{E}(\omega^\alpha) = \mathcal{E}(\alpha)$. $\mathcal{E}(\Omega_\alpha) = \mathcal{E}(\alpha)$.

2. $\mathcal{E}(\psi_\pi^{\vec{\nu}}(a)) = \{\psi_\pi^{\vec{\nu}}(a)\}$.

3. $\mathcal{A}(\alpha) = \bigcup\{\mathcal{A}(\beta) : \beta \in \mathcal{E}(\alpha)\}$ for $\mathcal{A} \in \{G_\kappa, F_\delta, k_\delta\}$.

4.

$$G_\kappa(\psi_\pi^{\vec{\nu}}(a)) = \begin{cases} G_\kappa(\{\pi, a\} \cup K^2(\vec{\nu})) & \kappa < \pi \\ G_\kappa(\pi) & \pi < \kappa \,\&\, \pi \not\preceq \kappa \\ \{\psi_\pi^{\vec{\nu}}(a)\} & \pi \preceq \kappa \end{cases}$$

$$F_\delta(\psi_\pi^{\vec{\nu}}(a)) = \begin{cases} F_\delta(\{\pi, a\} \cup K^2(\vec{\nu})) & \psi_\pi^{\vec{\nu}}(a) \geq \delta \\ \{\psi_\pi^{\vec{\nu}}(a)\} & \psi_\pi^{\vec{\nu}}(a) < \delta \end{cases}$$

$$k_\delta(\psi_\pi^{\vec{\nu}}(a)) = \begin{cases} \{\{\psi_\pi^{\vec{\nu}}(a)\}\} \cup k_\delta(\{\pi, a\} \cup K^2(\vec{\nu})) & \psi_\pi^{\vec{\nu}}(a) \geq \delta \\ \emptyset & \psi_\pi^{\vec{\nu}}(a) < \delta \end{cases}$$

For $\mathcal{A} \in \{K_\delta, G_\kappa, F_\delta, k_\delta\}$ and sets $X \subset OT$, $\mathcal{A}(X) := \bigcup\{\mathcal{A}(\alpha) : \alpha \in X\}$.

Let OT_n denote the subsystem of OT such that $\alpha \in OT_n$ iff each ordinal subterm occurring in α is smaller than $\omega_n(\mathbb{K}+1)$.

Definition 6.4 1. *For $\alpha \in OT \cap \mathbb{K}$, $\alpha \in OT_n \Leftrightarrow \mathcal{E}(\alpha) \subset OT_n$.*

2. *If $\alpha =_{NF} \omega^\beta < \omega_n(\mathbb{K}+1)$ with $\mathbb{K} < \beta \in OT_n$, then $\alpha \in OT_n$,*

3. Let $\alpha=\psi_{\pi}^{\vec{\nu}}(a)\in OT$ such that $\{a,\pi\}\cup K^{2}(\vec{\nu})\subset OT_{n}$. Then $\alpha=\psi_{\pi}^{\vec{\nu}}(a)\in OT_{n}$.

Proposition 6.5 *For any $n<\omega$ and $\delta=\psi_{\Omega}(\omega_{n}(\mathbb{K}+1))$,*

1. $\forall\alpha\in OT\cap\psi_{\Omega}(\omega_{n}(\mathbb{K}+1))(\alpha\in OT_{n})$.
2. $\forall\alpha\in OT(\{\alpha\}\cup K_{\delta}(\alpha)<\omega_{n}(\mathbb{K}+1)\Rightarrow\alpha\in OT_{n})$.

Proof. These are shown simultaneously by induction on $\ell\alpha$ for $\alpha\in OT$.

If α is not a strongly critical number, then IH yields the lemmas. Let $\alpha=\psi_{\pi}^{\vec{\nu}}(b)$ for some $\pi,\vec{\nu},b$.

6.5.1. Let $\alpha<\psi_{\Omega}(\omega_{n}(\mathbb{K}+1))$. Since Ω is the least recursively regular ordinal, $\psi_{\Omega}(\omega_{n}(\mathbb{K}+1))<\Omega$ and $K_{\alpha}(\{\Omega,\omega_{n}(\mathbb{K}+1)\})=\emptyset$, we see that $b<\omega_{n}(\mathbb{K}+1)$, $\psi_{\pi}^{\vec{\nu}}(b)<\Omega$ and $K_{\delta}(K^{2}(\vec{\nu})\cup\{\pi,b\})<\omega_{n}(\mathbb{K}+1)$. By $\psi_{\pi}^{\vec{\nu}}(b)<\Omega$ we obtain $\pi=\Omega$, and $\vec{\nu}=\vec{0}$. IH on Proposition 6.5.1 with $\{b\}\cup K_{\delta}(b)<\omega_{n}(\mathbb{K}+1)$ yields $b\in OT_{n}$.

6.5.2. Let $K_{\delta}(\alpha)<\omega_{n}(\mathbb{K}+1)$. If $\alpha<\delta=\psi_{\Omega}(\omega_{n}(\mathbb{K}+1))$, then $\alpha\in OT_{n}$ by Proposition 6.5.1. Suppose $\alpha\geq\delta$. Then $K_{\delta}(\alpha)=\{b\}\cup K_{\delta}(K(\vec{\nu})\cup\{\pi,b\})$. IH with $\pi\leq\mathbb{K}$ yields $\{b,\pi\}\subset OT_{n}$. In particular $b<\omega_{n}(\mathbb{K}+1)$. This yields $K(\vec{\nu})\leq b<\omega_{n}(\mathbb{K}+1)$ by Definition 3.3.14 and 3.3.15. Hence by IH we obtain $K(\vec{\nu})\subset OT_{n}$. □

Therefore it suffices show the following Theorem 6.6 to prove Theorem 1.2.

Theorem 6.6 *For* each $n<\omega$, $\mathsf{KP}\Pi_{N}$ *proves that* $(OT_{n},<)$ *is well-founded.*

Definition 6.7 $S(\eta)$ denotes the set of immediate subterms of η when $\eta\not\in\mathcal{E}(\eta)$. For example $S(\varphi\beta\gamma)=\{\beta,\gamma\}$. $S(0):=S(\mathbb{K}):=\emptyset$ and $S(\eta)=\{\eta\}$ when $\eta\in\mathcal{E}(\eta)$.

Proposition 6.8 *For $\alpha,\kappa,a,b\in OT$,*

1. $G_{\kappa}(\alpha)\leq\alpha$.
2. $\alpha\in\mathcal{H}_{a}(b)\Rightarrow G_{\kappa}(\alpha)\subset\mathcal{H}_{a}(b)$.
3. *Let $\gamma\leq\delta$. Then $F_{\gamma}(\alpha)<\beta\,\&\,F_{\delta}(\alpha)<\gamma\Rightarrow F_{\delta}(\alpha)<\beta$.*

Proof by simultaneous induction on $\ell\alpha$. It is easy to see that

$$G_{\kappa}(\alpha)\ni\beta\Rightarrow\beta\prec\kappa\,\&\,\ell\kappa<\ell\beta\leq\ell\alpha \tag{38}$$

6.8.1. Consider the case $\alpha=\psi_{\pi}^{\vec{\nu}}(a)$ with $\pi\not\preceq\kappa$. First let $\kappa<\pi$. Then $G_{\kappa}(\alpha)=G_{\kappa}(\{\pi,a\}\cup K^{2}(\vec{\nu}))$. On the other hand we have $\forall\gamma\in K^{2}(\vec{\nu})\cup\{\pi,a\}(K_{\alpha}(\gamma)<a)$, i.e, $K(\vec{\nu})\cup\{\pi,a\}\subset\mathcal{H}_{a}(\alpha)$. Proposition 6.8.2 with (38) yields $G_{\kappa}(K^{2}(\vec{\nu})\cup\{\pi,a\})\subset\mathcal{H}_{a}(\alpha)\cap\kappa\subset\mathcal{H}_{a}(\alpha)\cap\pi\subset\alpha$. Hence $G_{\kappa}(\alpha)<\alpha$.

Next let $\pi<\kappa$ and $\pi\not\preceq\kappa$. Then $G_{\kappa}(\alpha)=G_{\kappa}(\pi)$. By IH we have $G_{\kappa}(\pi)\leq\pi$, and $G_{\kappa}(\pi)<\pi$ by $\pi\not\preceq\kappa$. On the other hand we have $K_{\alpha}(\pi)<a$, i.e, $\pi\in\mathcal{H}_{a}(\alpha)$.

Proposition 6.8.2 yields $G_\kappa(\pi) \subset \mathcal{H}_a(\alpha) \cap \pi \subset \alpha$. Hence $G_\kappa(\alpha) < \alpha$.

6.8.2. Since $G_\kappa(\alpha) \leq \alpha$ by Proposition 6.8.1, we can assume $\alpha \geq b$. Again consider the case $\alpha = \psi_\pi^{\vec{\nu}}(a)$ with $\pi \not\preceq \kappa$. Then $K^2(\vec{\nu}) \cup \{\pi, a\} \subset \mathcal{H}_a(b)$ and $G_\kappa(\alpha) \subset G_\kappa(K^2(\vec{\nu}) \cup \{\pi, a\})$. IH yields the lemma.

6.8.3. This is seen by induction on $\ell\alpha$. □

Proposition 6.9 *Let $\beta \preceq \alpha = \psi_\pi^{\vec{\nu}}(a)$. Then $F_\pi(K^2(\vec{\nu})) < \beta$.*

Proof. Let $pd^{(i-1)}(\beta) = \pi_{i-1} = \psi_{\pi_i}^{\vec{\nu}_i}(a_i)$ with $\beta = \pi_0$ and $\pi = \pi_n$. Then by $\pi_{i-1} < \pi_i$ we have $\pi_i \in \mathcal{H}_{a_{j+1}}(\pi_j)$ for any $j < i$, and $K^2(\vec{\nu}) \subset \mathcal{H}_{a_{j+1}}(\pi_j)$ for $\vec{\nu} = \vec{\nu}_n$ and any $j < n$. On the other hand we have $\mathcal{H}_{a_{j+1}}(\pi_j) \cap \pi_{j+1} \subset \pi_j$. We see by induction on $n - j \geq 0$ that $F_\pi(K^2(\vec{\nu})) < \pi_j$. □

Proposition 6.10 *Let $\gamma \preceq \tau$ and $\gamma \not\prec \kappa$. Then $G_\kappa(\tau) \subset G_\kappa(\gamma)$.*

Proof. Let $\gamma \not\prec \kappa$. We show $\gamma \preceq \tau \Rightarrow G_\kappa(\tau) \subset G_\kappa(\gamma)$ by induction on $\ell\gamma - \ell\tau$. Let $\gamma \preceq \tau = \psi_\pi^{\vec{\nu}}(a)$. By IH we have $G_\kappa(\tau) \subset G_\kappa(\gamma)$. On the other hand we have $G_\kappa(\pi) \subset G_\kappa(\tau)$ since $\pi \not\prec \kappa$ and $\pi = \kappa \Rightarrow G_\kappa(\pi) = \emptyset$, cf. (38). □

Proposition 6.11 *Let $a, \alpha, \kappa, \beta, \delta \in T$ with $\alpha = \psi_\pi^{\vec{\nu}}(a)$ for some $\{a\} \cup K^2(\vec{\nu}) \subset T$. If $\beta \notin \mathcal{H}_a(\alpha)$ and $K_\delta(\beta) < a$, then there exists a $\gamma \in F_\delta(\beta)$ such that $\mathcal{H}_a(\alpha) \not\ni \gamma < \delta$.*

Proof. By induction on $\ell\beta$. Assume $\beta \notin \mathcal{H}_a(\alpha)$ and $K_\delta(\beta) < a$. By IH we can assume that $\beta = \psi_\kappa^{\vec{\xi}}(b)$. If $\beta < \delta$, then $\beta \in F_\delta(\beta)$, and $\gamma = \beta$ is a desired one. Assume $\beta \geq \delta$. Then we have $K_\delta(\beta) = \{b\} \cup K_\delta(\{b, \kappa\} \cup K^2(\vec{\xi})) < a$. In particular $b < a$, and hence $\{b, \kappa\} \cup K^2(\vec{\xi}) \not\subset \mathcal{H}_a(\alpha)$. By IH there exists a $\gamma \in F_\delta(\{b, \kappa\} \cup K^2(\vec{\xi})) = F_\delta(\beta)$ such that $\mathcal{H}_a(\alpha) \not\ni \gamma < \delta$. □

6.2 Rudiments of distinguished sets

In this subsection, working in the set theory KPℓ for limits of admissibles, we will develop rudiments of distinguished classes, which was first introduced by W. Buchholz [10]. Since many properties of distingusihed classes are seen as in [2, 4], we will omit their proofs.

As in [4] our welfoundedness proof inside KPΠ_N goes as follows. The wellfoundedness of OT is reduced to one of the relation $\prec$ in the following way. $\alpha \in V(X)$ in Definition 6.12.3 is intended for α to be in the wellfounded part of $\prec$ with respect to a set X. In Lemma 6.38 it is shown for a Δ_1-class $\mathcal{G}(X)$ defined in Definition 6.26, that $\eta \in \mathcal{G}(X) \cap V(X)$ yields the existence of a distinguished set X' such that $\eta \in X'$ *provided that* X is a distinguished set which is closed under the 'hyperjump' operation $X \mapsto X'$ for any $\gamma \prec \eta$. Let us call such an X *η-Mahlo*. It turns out that we need the fact that $X \subset V(X)$ for any distinguished sets X in proving Lemma 6.38. Furthermore we need even stronger

condition $X \subset V^*(X)$ for the Claim 6.39 in Lemma 6.38, where $V^*(X)$ is defined in Definition 6.12.5. This motivates our Definition 6.19.1 of distinguished sets (41).

There remain three tasks for each $\eta \in OT$. One is to show that $\eta \in \mathcal{G}(X)$, second to show $\eta \in V(X)$, and third the existence of an η-Mahlo distinguished set. It is not hard to show $\eta \in \mathcal{G}(X)$ by induction on a for $\eta = \psi_\pi^{\vec{\nu}}(a)$, cf. Lemma 8.2. Next for sets P let $\mathcal{W}^P$ be the maximal distinguished class *in* P. $\mathcal{W}^P$ is Σ_1^P, i.e., Σ_1-definable class on P, and $\mathcal{W}^Q$ is a distinguished *set* in P for any sets $Q \in P$, cf. subsection 6.4. In particular $\mathcal{W} = \mathcal{W}^{\mathrm{L}}$ is the maximal distinguished class for the whole Π_N-reflecting universe L. Let us say that P is η-Mahlo if $\mathcal{W}^P$ is an η-Mahlo distinguished class. In view of Lemma 6.38 P is η-Mahlo if P is Π_2-reflecting on γ-Mahlo sets for any $\gamma \prec \eta$ since $\mathcal{G}(\mathcal{W}^P)$ is Π_2^P. This means that we need to iterate recursively Mahlo operations along $\prec$ up to a given η assuming that η is in the wellfounded part $V(\mathcal{W})$. Now if $\gamma \prec \eta$, then the sequence of ordinals $\{m_k(\gamma)\}_k$ is smaller than $\{m_k(\eta)\}_k$ in a sense. Indeed we could assign an ordinal $o_1(\{m_k(\gamma)\}) < \varepsilon_{\mathbb{K}+2}$ in such a way that $o_1(\{m_k(\gamma)\}) < o_1(\{m_k(\eta)\})$ as in Definition 2.14. However if we refer such a big ordinal $o_1(\{m_k(\eta)\}) > \eta$ explicitly in defining η to be in $V(\mathcal{W})$, the persistency (39) in Definition 6.12.3 does not hold. As we see it in this section, the persistency is crucial for distinguished sets, cf. Proposition 6.23.

The k-predecessors defined in subsection 7.1 are needed for us to embed the relation $\prec$ on OT to an exponential structure induced solely from ordinals $\{m_k(\eta)\}_k$, cf. Lemma 7.21. Coefficients in the exponential structure for η consist of hereditary k-predecessors of η for $2 \leq k \leq N-1$. Roughly $\eta \in V(X) = V_N(X)$ introduced in subsection 7.3 if these coefficients are in the wellfounded part of some relations $<_k$. For the persistency (39) of $V_N(X)$ we need to augment a datum to the exponential structure. Then $\mathrm{KP}\Pi_N$ proves the existence of an η-Mahlo universe under the condition $\eta \in V_N(\mathcal{W})$, cf. Theorem 7.5 and Lemma 7.24.2. On the other side relations $<_k$ are defined so that $\beta <_k \gamma \Rightarrow st(m_k(\beta)) < st(m_k(\gamma))$. Hence the task to show $\eta \in V_N(\mathcal{W})$ is reduced to show the wellfoundedness of $<$ on OT, cf. Lemma 8.8. Thus these three tasks together with showing the wellfoundedness of $<$ have to be done simultaneously.

$X, Y, \ldots$ range over *subsets* of OT_n. While $\mathcal{X}, \mathcal{Y}, \ldots$ range over *classes*.

We define sets $\mathcal{C}^\alpha(X) \subset OT_n$ for $\alpha \in OT_n, X \subset OT_n$ as follows.

Definition 6.12 Let $\alpha, \beta \in OT_n, X \subset OT_n$.

1. Let $\mathcal{C}^\alpha(X)$ be the closure of $\{0, \mathbb{K}\} \cup (X \cap \alpha)$ under $+$, $\mathbb{K} < \beta \mapsto \omega^\beta \in OT_n$, $(\beta, \gamma) \mapsto \varphi\beta\gamma\,(\beta, \gamma < \mathbb{K})$, $\mathbb{K} > \beta \mapsto \Omega_\beta > \beta$, and $(\sigma, \beta, \vec{\xi}) \mapsto \psi_\sigma^{\vec{\xi}}(\beta)$ for $\sigma > \alpha$ in OT_n.

   The last clauses say that, if $\Omega_\beta > \beta \in \mathcal{C}^\alpha(X) \Rightarrow \Omega_\beta \in \mathcal{C}^\alpha(X)$, and $\psi_\sigma^{\vec{\xi}}(a) \in \mathcal{C}^\alpha(X)$ if $\{\sigma, a\} \cup K^2(\vec{\xi}) \subset \mathcal{C}^\alpha(X)$ and $\sigma > \alpha$.

2. $\alpha^+ = \Omega_{a+1}$ denotes the least recursively regular term above α if such a term exists. Otherwise $\alpha^+ := \infty$. Obviously α^+ is computable from α.

3. $V(X)$ is a Δ_1-class such that

$$
\begin{array}{rl}
& \forall\alpha < \mathbb{K}[(X \cap \alpha = Y \cap \alpha \Rightarrow V(X) \cap \alpha^+ = V(Y) \cap \alpha^+) \\
\land & (\neg\exists\kappa, a, \vec{\xi} \neq \vec{0}(\alpha =_{NF} \psi_\kappa^{\vec{\xi}}(a)) \Rightarrow \alpha \in V(X))]
\end{array}
\tag{39}
$$

4. $V\mathcal{C}^\alpha(X) := V(X) \cap \mathcal{C}^\alpha(X)$.

5. $\alpha \in V^*(X) :\Leftrightarrow \alpha \in V(X) \,\&\, \mathcal{C}^\alpha(X) \cap \alpha \subset V(X)$.

6. $V^*\mathcal{C}^\alpha(X) := V^*(X) \cap \mathcal{C}^\alpha(X)$.

Proposition 6.13 *$X \cap \alpha = Y \cap \alpha \Rightarrow \mathcal{C}^\alpha(X) = \mathcal{C}^\alpha(Y)$ and $X \mapsto \mathcal{C}^\alpha(X)$ is monotonic.*

Proposition 6.14 *$\alpha < \beta < \alpha^+ \Rightarrow \mathcal{C}^\alpha(X) \subset \mathcal{C}^\beta(X)$.*

Proof. By induction on $\ell\gamma\,(\gamma \in OT_n)$ we see that $\gamma \in \mathcal{C}^\alpha(X) \Rightarrow \gamma \in \mathcal{C}^\beta(X)$. □

Proposition 6.15 *Let $\delta \leq \mathbb{K}$. Then $F_\delta(\alpha) \cup k_\delta(\alpha) \subset X \Rightarrow \alpha \in \mathcal{C}^{\mathbb{K}}(X)$.*

Proof. This is seen by induction on $\ell\alpha$. □

Proposition 6.16 *Assume $\alpha \in \mathcal{C}^\alpha(X)$ and $\alpha \preceq \sigma$. Then $\sigma \in \mathcal{C}^\alpha(X)$.*

Proof. We see by induction on $\ell\alpha - \ell\sigma$ that $\alpha \in \mathcal{C}^\alpha(X) \,\&\, \alpha \preceq \sigma \Rightarrow \sigma \in \mathcal{C}^\alpha(X)$. □

Proposition 6.17 (Cf. [2], Lemmas 3.5.3 and 3.5.4.) *Assume $\forall\gamma \in X[\gamma \in \mathcal{C}^\gamma(X)]$ for a set $X \subset OT_n$.*

1. *$\alpha \leq \beta \Rightarrow \mathcal{C}^\beta(X) \subset \mathcal{C}^\alpha(X)$.*

2. *$\alpha < \beta < \alpha^+ \Rightarrow \mathcal{C}^\beta(X) = \mathcal{C}^\alpha(X)$.*

Proof. 6.17.1. We see by induction on $\ell\gamma\,(\gamma \in OT_n)$ that

$$
\forall\beta \geq \alpha[\gamma \in \mathcal{C}^\beta(X) \Rightarrow \gamma \in \mathcal{C}^\alpha(X) \cup (X \cap \beta)]
\tag{40}
$$

For example, if $\psi_\pi^{\vec{\xi}}(\delta) \in \mathcal{C}^\beta(X)$ with $\pi > \beta \geq \alpha$ and $\{\pi, \delta\} \cup K^2(\vec{\xi}) \subset \mathcal{C}^\alpha(X) \cup (X \cap \beta)$, then $\pi \in \mathcal{C}^\alpha(X)$, and for any $\gamma \in \{\delta\} \cup K^2(\vec{\xi})$, either $\gamma \in \mathcal{C}^\alpha(X)$ or $\gamma \in X \cap \beta$ by IH. If $\gamma < \alpha$, then $\gamma \in X \cap \alpha \subset \mathcal{C}^\alpha(X)$. If $\alpha \leq \gamma \in X \cap \beta$, then $\gamma \in \mathcal{C}^\gamma(X)$ by the assumption, and by IH we have $\gamma \in \mathcal{C}^\alpha(X) \cup (X \cap \gamma)$, i.e., $\gamma \in \mathcal{C}^\alpha(X)$. Therefore $\{\pi, \delta\} \cup K^2(\vec{\xi}) \subset \mathcal{C}^\alpha(X)$, and $\psi_\pi^{\vec{\xi}}(\delta) \in \mathcal{C}^\alpha(X)$.

Using (40) we see from the assumption that $\forall\beta \geq \alpha[\gamma \in \mathcal{C}^\beta(X) \Rightarrow \gamma \in \mathcal{C}^\alpha(X)]$.

6.17.2. Assume $\alpha < \beta < \alpha^+$. Then by Proposition 6.17.1 we have $\mathcal{C}^\beta(X) \subset \mathcal{C}^\alpha(X)$. Conversely $\mathcal{C}^\alpha(X) \subset \mathcal{C}^\beta(X)$ is seen from Proposition 6.14. □

Definition 6.18 *1.* $Prg[X, Y] :\Leftrightarrow \forall\alpha \in X(X \cap \alpha \subset Y \to \alpha \in Y)$.

2. For a definable class $\mathcal{X}$, $TI[\mathcal{X}]$ denotes the schema:
$TI[\mathcal{X}] :\Leftrightarrow Prg[\mathcal{X}, \mathcal{Y}] \to \mathcal{X} \subset \mathcal{Y}$ holds for *any definable class* $\mathcal{Y}$.

3. For $X \subset OT_n$, $W(X)$ denotes the *wellfounded part* of X.

4. $Wo[X] :\Leftrightarrow X \subset W(X)$.

Note that for $\alpha \in OT_n$, $W(X) \cap \alpha = W(X \cap \alpha)$.

Definition 6.19 For $X \subset OT_n$ and $\alpha \in OT_n$,

1.
$$D[X] :\Leftrightarrow X < \mathbb{K} \,\&\, \forall\alpha(\alpha \leq X \to W(V^*\mathcal{C}^\alpha(X)) \cap \alpha^+ = X \cap \alpha^+) \quad (41)$$

A class $\mathcal{X}$ is said to be a *distinguished class* if $D[\mathcal{X}]$. A *distinguished set* is a set which is a distinguished class.

2. $\mathcal{W} := \bigcup\{X : D[X]\}$.

Since, in KPℓ, the wellfounded part $W(X)$ of a set X is again a set, $D[X]$ is Δ_1. Hence both $\mathcal{W}$ and $\mathcal{C}^\alpha(\mathcal{W})$ are Σ_1. Obviously any distinguished set X enjoys the condition $\forall\alpha \in X[\alpha \in V^*\mathcal{C}^\alpha(X)]$.

Proposition 6.20 $D[X] \Rightarrow Wo[X]$.

Proposition 6.21 (Cf. Lemma 3.30 in [4].)
Let X be a distinguished set. Then $\alpha \in X \Rightarrow \forall\beta[\alpha \in \mathcal{C}^\beta(X)]$.

Proposition 6.22 (Cf. Lemma 3.28 in [4].)
For any distinguished sets X and Y, the following holds:

$$X \cap \alpha = Y \cap \alpha \Rightarrow \forall\beta < \alpha^+\{V^*\mathcal{C}^\beta(X) \cap \beta^+ = V^*\mathcal{C}^\beta(Y) \cap \beta^+\}.$$

Proof. Assume that $X \cap \alpha = Y \cap \alpha$ and $\beta < \alpha^+$. By the condition (39) we have $V(X) \cap \beta^+ = V(Y) \cap \beta^+$.

On the other hand we have by Propositions 6.17.2 and 6.13, $\mathcal{C}^\beta(X) = \mathcal{C}^\beta(Y)$, and for any $\delta < \beta^+$, $\mathcal{C}^\delta(X) = \mathcal{C}^\delta(Y)$. Hence $V^*(X) \cap \beta^+ = V^*(Y) \cap \beta^+$. □

Proposition 6.23 *Let X and Y be distinguished sets.*

1. *$\alpha \leq X \,\&\, \alpha \leq Y \Rightarrow X \cap \alpha^+ = Y \cap \alpha^+$.*

2. *Either $X \subset_e Y$ or $Y \subset_e X$, where $X \subset_e Y$ designates that Y is an end extension of X, i.e., $X \subset_e Y :\Leftrightarrow X \subset Y \,\&\, \forall\alpha \in Y \forall\beta \in X(\alpha < \beta \to \alpha \in X)$.*

Proposition 6.24 *$\mathcal{W}$ is the maximal distinguished class, i.e., $D[\mathcal{W}]$. Also $TI[\mathcal{W}]$ for $\mathcal{W} \subset \mathbb{K}$.*

6.3 Sets $\mathcal{C}^\alpha(X)$ and $\mathcal{G}(X)$

In this subsection we will establish elementary properties on sets $\mathcal{C}^\alpha(X)$.

Proposition 6.25 (Cf. [2], Lemma 3.6.) *Let $\gamma < \beta$. For a distinguished set X assume $\alpha \in \mathcal{C}^\gamma(X)$ and $\forall\kappa \leq \beta[G_\kappa(\alpha) < \gamma]$.*

1. *Assume* LIH : $\forall\delta[\ell\delta \leq \ell\alpha \,\&\, \delta \in \mathcal{C}^\gamma(X) \cap \gamma \Rightarrow \delta \in \mathcal{C}^\beta(X)]$. *Then $\alpha \in \mathcal{C}^\beta(X)$.*
2. $\mathcal{C}^\gamma(X) \cap \gamma \subset X \Rightarrow \alpha \in \mathcal{C}^\beta(X)$.

Proof. 6.25.1 by induction on $\ell\alpha$. If $\alpha < \gamma$, then $\alpha \in \mathcal{C}^\gamma(X) \cap \gamma$. LIH yields $\alpha \in \mathcal{C}^\beta(X)$. Assume $\alpha \geq \gamma$. Except the case $\alpha = \psi_\pi^{\vec{\nu}}(a)$ for some $\pi, a, \vec{\nu}$, IH yields $\alpha \in \mathcal{C}^\beta(X)$. Suppose $\alpha = \psi_\pi^{\vec{\nu}}(a)$ for some $\{\pi, a\} \cup K^2(\vec{\nu}) \subset \mathcal{C}^\gamma(X)$ and $\pi > \gamma$. If $\pi \leq \beta$, then $\{\alpha\} = G_\pi(\alpha) < \gamma$ by the second assumption. Hence this is not the case, and we obtain $\pi > \beta$. Then $G_\kappa(\{\pi, a\} \cup K^2(\vec{\nu})) = G_\kappa(\alpha) < \gamma$ for any $\kappa \leq \beta < \pi$. IH yields $\{\pi, a\} \cup K^2(\vec{\nu}) \subset \mathcal{C}^\beta(X)$. We conclude $\alpha \in \mathcal{C}^\beta(X)$ from $\pi > \beta$.

$\Box$

Definition 6.26 $\mathcal{G}(X) := \{\alpha : \alpha \in \mathcal{C}^\alpha(X) \,\&\, \mathcal{C}^\alpha(X) \cap \alpha \subset X\}$.

Proposition 6.27 *Let $\alpha \in \mathcal{C}^\beta(X)$ and $X \cap \beta \subset \mathcal{G}(X)$ for a distinguished set X. Assume $X \cap \beta < \delta$. Then $F_\delta(\alpha) \subset \mathcal{C}^\beta(X)$.*

Proof. By induction on $\ell\alpha$. Let $\{0, \mathbb{K}\} \not\ni \alpha \in \mathcal{C}^\beta(X)$. First consider the case $\alpha \not\in \mathcal{E}(\alpha)$. If $\alpha \in X \cap \beta \subset \mathcal{G}(X)$, then $\mathcal{E}(\alpha) \subset \mathcal{C}^\alpha(X) \cap \alpha \subset X \subset \mathcal{C}^\beta(X)$ by Proposition 6.21. Otherwise we have $\alpha \not\in \mathcal{E}(\alpha) \subset \mathcal{C}^\beta(X)$. In each case IH yields $F_\delta(\alpha) = F_\delta(\mathcal{E}(\alpha)) \subset \mathcal{C}^\beta(X)$.

Let $\alpha = \psi_\pi^{\vec{\nu}}(a)$ for some $\pi, \vec{\nu}, a$. If $\alpha < \delta$, then $F_\delta(\alpha) = \{\alpha\}$, and there is nothing to prove. Let $\alpha \geq \delta$. Then $F_\delta(\alpha) = F_\delta(\{\pi, a\} \cup K^2(\vec{\nu}))$. On the other side we see $\{\pi, a\} \cup K^2(\vec{\nu}) \subset \mathcal{C}^\beta(X)$ from $\alpha \in \mathcal{C}^\beta(X)$ and the assumption. IH yields $F_\delta(\alpha) \subset \mathcal{C}^\beta(X)$. $\Box$

Next we show $X \subset \mathcal{G}(X)$ for any distinguished set X, cf. Lemma 6.31.

Proposition 6.28 *Let X be a distinguished set, and assume $X \cap \beta \subset \mathcal{G}(X)$.*

1. $\forall\tau[\alpha \in X \cap \beta \Rightarrow G_\tau(\alpha) \subset X]$.
2. $\forall\beta\forall\tau[\alpha \in \mathcal{C}^\beta(X) \Rightarrow G_\tau(\alpha) \subset \mathcal{C}^\beta(X)]$.

Proof. By simultaneous induction on $\ell\alpha$.
6.28.1. Suppose $\alpha \in X \cap \beta \subset \mathcal{G}(X)$. Then $\alpha \in \mathcal{C}^\alpha(X)$, and $\mathcal{C}^\alpha(X) \cap \alpha \subset X$.

Let $\alpha \not\in \mathcal{E}(\alpha)$. Then $\mathcal{E}(\alpha) \subset \mathcal{C}^\alpha(X) \cap \alpha \subset X$. IH yields $G_\tau(\alpha) = G_\tau(\mathcal{E}(\alpha)) \subset X$. Assume $\alpha \in \mathcal{E}(\alpha)$, i.e., $\alpha = \psi_\pi^{\vec{\nu}}(a)$ for some $\pi, a, \vec{\nu}$. Then $\{\pi, a\} \cup K(\vec{\nu}) \subset \mathcal{C}^\alpha(X)$ by $\alpha \in \mathcal{C}^\alpha(X)$. We can assume $\pi \not\leq \tau$. Then $G_\tau(\alpha) \subset G_\tau(\{\pi, a\} \cup K^2(\vec{\nu}))$. By IH with Proposition 6.8.1 we have $G_\tau(\alpha) \subset G_\tau(\{\pi, a\} \cup K^2(\vec{\nu})) \subset \mathcal{C}^\alpha(X) \cap$

$\alpha \subset X$.

6.28.2. Assume $\alpha \in \mathcal{C}^{\beta}(X)$. We show $G_{\tau}(\alpha) \subset \mathcal{C}^{\beta}(X)$. If $\alpha \in X \cap \beta$, then by Proposition6.28.1 we have $G_{\tau}(\alpha) \subset X \cap \beta \subset \mathcal{C}^{\beta}(X)$. Consider the case $\alpha \not\in X \cap \beta$. If $\alpha \not\in \mathcal{E}(\alpha)$, then IH yields $G_{\tau}(\alpha) = G_{\tau}(\mathcal{E}(\alpha)) \subset \mathcal{C}^{\beta}(X)$. Let $\alpha = \psi_{\pi}^{\vec{\nu}}(a)$ for some $\{\pi, a\} \cup K^2(\vec{\nu}) \subset \mathcal{C}^{\beta}(X)$ with $\beta < \pi \not\preceq \tau$. IH yields $G_{\tau}(\alpha) \subset G_{\tau}(\{\pi, a\} \cup K^2(\vec{\nu})) \subset \mathcal{C}^{\beta}(X)$. $\Box$

Proposition 6.29 *Let X be a distinguished set, and assume $X \cap \beta \subset \mathcal{G}(X)$. Then*

$$\forall\alpha\forall\sigma \leq \beta[\alpha \in \mathcal{C}^{\beta}(X) \Rightarrow G_{\sigma}(\alpha) \subset X].$$

Proof. By induction on $\ell\alpha$ using Proposition 6.28.1 we see $\alpha \in \mathcal{C}^{\beta}(X) \,\&\, \sigma \leq \beta \Rightarrow G_{\sigma}(\alpha) \subset X$. $\Box$

Proposition 6.30 *Let X be a distinguished set. Assume $X \cap \gamma \subset \mathcal{G}(X)$, and $\alpha \in \mathcal{C}^{\gamma}(X) \cap \gamma$. Then $\mathcal{C}^{\alpha}(X) \cap \alpha \subset \mathcal{C}^{\gamma}(X)$.*

Proof. First suppose that there exists a δ such that $\alpha \leq \delta \in X \cap \gamma \subset \mathcal{G}(X)$. Then $\mathcal{C}^{\delta}(X) \cap \delta \subset X$. If $\alpha = \delta$, then $\mathcal{C}^{\alpha}(X) \cap \alpha \subset X \subset \mathcal{C}^{\gamma}(X)$ by Proposition 6.21. Let $\alpha < \delta$. Then $X \cap \delta \subset \mathcal{G}(X)$, and $\alpha \in \mathcal{C}^{\delta}(X) \cap \delta$ by Proposition 6.17.1. Moreover we have $\delta \in X$. Therefore it suffices to show the proposition under the *assumption* $\gamma \in X$, for then $\mathcal{C}^{\alpha}(X) \cap \alpha \subset \mathcal{C}^{\delta}(X) \cap \alpha \subset X \subset \mathcal{C}^{\gamma}(X)$.

Let us prove the proposition by main induction on $\gamma \in X$. If $\alpha \leq X \cap \gamma$, then MIH yields the proposition as we saw it above. In what follows assume $X \cap \gamma < \alpha$.

By subsidiary induction on $\ell\alpha + \ell\beta$ we show that

$$\beta \in \mathcal{C}^{\alpha}(X) \cap \alpha \Rightarrow \beta \in \mathcal{C}^{\gamma}(X).$$

If $\beta \in X$, then $\beta \in \mathcal{C}^{\gamma}(X)$ follows from Proposition 6.21. In what follows suppose $\beta \not\in X$

If $\beta \not\in \mathcal{E}(\beta)$, then $\beta \in \mathcal{C}^{\gamma}(X)$ is seen from SIH. Assume $\beta = \psi_{\pi}^{\vec{\nu}}(a)$ with a $\pi > \alpha$ and some $K^2(\vec{\nu}) \cup \{\pi, a\} \subset \mathcal{C}^{\alpha}(X)$. If $\alpha \not\in \mathcal{E}(\alpha)$, then $\beta \leq \delta$ for some $\delta \in \mathcal{E}(\alpha) \subset \mathcal{C}^{\gamma}(X) \cap \gamma$. Since $\ell\delta < \ell\alpha$, SIH yields $\beta \in \mathcal{C}^{\gamma}(X)$. Let $\alpha = \psi_{\kappa}^{\vec{\xi}}(b)$ for some $\kappa, b, \vec{\xi}$. By $X \not\ni \alpha \in \mathcal{C}^{\gamma}(X)$ we have $\gamma < \kappa$.

First consider the case $\gamma < \pi$. Then $\forall\sigma \leq \gamma[G_{\sigma}(\{\pi, a\} \cup K^2(\vec{\nu})) = G_{\sigma}(\beta) < \beta < \alpha]$ by Proposition 6.8.1. Since $\ell\eta < \ell\beta$ for each $\eta \in \{\pi, a\} \cup K^2(\vec{\nu})$, by SIH we have LIH: $\forall\delta[\ell\delta \leq \ell\eta \,\&\, \delta \in \mathcal{C}^{\alpha}(X) \cap \alpha \Rightarrow \delta \in \mathcal{C}^{\gamma}(X)]$ in Proposition 6.25.1, which yields $\{\pi, a\} \cup K^2(\vec{\nu}) \subset \mathcal{C}^{\gamma}(X)$, and $\beta \in \mathcal{C}^{\gamma}(X)$.

Next assume $\pi \leq \gamma < \kappa$. $\pi \not\in \mathcal{H}_b(\alpha)$ since otherwise by $\pi < \kappa$ we would have $\pi < \alpha$. Then by Proposition 2.17 we have $a \geq b$ and $K^2(\vec{\xi}) \cup \{\kappa, b\} \not\subset \mathcal{H}_a(\beta)$. On the other hand we have $K_{\alpha}(K^2(\vec{\xi}) \cup \{\kappa, b\}) < b \leq a$. By Proposition 6.11 pick a $\delta \in F_{\alpha}(K^2(\vec{\xi}) \cup \{\kappa, b\})$ such that $\mathcal{H}_a(\beta) \not\ni \delta < \alpha$. We have $\ell\delta < \ell\alpha$ and $K^2(\vec{\xi}) \cup \{\kappa, b\} \subset \mathcal{C}^{\gamma}(X)$. Hence by Proposition 6.27 we obtain $\delta \in \mathcal{C}^{\gamma}(X) \cap \gamma$. From $\beta \not\in \mathcal{H}_a(\beta)$ we see $\beta \leq \delta$. If $\beta = \delta \in \mathcal{C}^{\gamma}(X)$, we are done. Let $\beta < \delta$. Then $\beta \in \mathcal{C}^{\delta}(X) \cap \delta$, and SIH with $\ell\delta < \ell\alpha$ yields $\beta \in \mathcal{C}^{\gamma}(X)$. $\Box$

Lemma 6.31 *Let X be a distinguished set. Then $X \subset \mathcal{G}(X)$, $\forall\alpha \in X \forall\tau(G_\tau(\alpha) \subset X)$, and $\forall\alpha \in \mathcal{C}^\gamma(X) \cap \gamma(\mathcal{C}^\alpha(X) \cap \alpha \subset \mathcal{C}^\gamma(X))$.*

Proof. We have $\gamma \in \mathcal{C}^\gamma(X)$ for $\gamma \in X$.

Assume $\alpha \in \mathcal{C}^\gamma(X) \cap \gamma$. We have $\gamma \in W(\mathcal{C}^\gamma(X)) \cap \gamma^+ = X \cap \gamma^+$ by $\gamma \in X$. Hence $\alpha \in W(\mathcal{C}^\gamma(X)) \cap \gamma^+ \subset X$. Next $\forall\alpha \in X\forall\tau(G_\tau(\alpha) \subset X)$ is seen from $X \subset \mathcal{G}(X)$ and Proposition 6.28.1. Finally $\forall\alpha \in \mathcal{C}^\gamma(X) \cap \gamma(\mathcal{C}^\alpha(X) \cap \alpha \subset \mathcal{C}^\gamma(X))$ is seen from Proposition $\Box$

The following Propositions 6.32 and 6.33 are seen from Lemma 6.31.

Proposition 6.32 *Let X be a distinguished set. Then $\alpha \leq X \cap \beta \,\&\, \alpha \in \mathcal{C}^\beta(X) \Rightarrow \alpha \in X$.*

Proposition 6.33 *Let X be a distinguished set, and $\alpha \in X$. Then $\mathcal{C}^\alpha(X) \cap \alpha \subset X$.*

Proposition 6.34 *Let X be a distinguished set, and $\alpha \in X$. Then $S(\alpha) \subset X$.*

Proof. Let $\alpha \in X$. Then $\alpha \in \mathcal{C}^\alpha(X)$ by Proposition 6.21. Hence $S(\alpha) \cap \alpha \subset \mathcal{C}^\alpha(X) \cap \alpha \subset X$ by Proposition 6.33. $\Box$

Proposition 6.35 *Let X be a distinguished set. $\alpha \in \mathcal{C}^\delta(X) \Rightarrow F_\delta(\alpha) \subset X$.*

Proof by induction on $\ell\alpha$. If $\alpha \in X \cap \delta$, then $S(\alpha) \subset X$ by Proposition 6.34, and $F_\delta(\alpha) = F_\delta(S(\alpha)) \subset X$ by IH. Otherwise $S(\alpha) \subset \mathcal{C}^\delta(X)$, and $F_\delta(\alpha) = F_\delta(S(\alpha)) \subset X$ by IH. $\Box$

Proposition 6.36 *Let X be a distinguished set, and put $Y = W(V^*\mathcal{C}^\alpha(X)) \cap \alpha^+$ for an $\alpha < \mathbb{K}$. Assume that $\alpha \in \mathcal{G}(X)$ and*

$$\forall\beta < \mathbb{K}(X < \beta \,\&\, \beta^+ < \alpha^+ \Rightarrow W(V^*\mathcal{C}^\beta(X)) \cap \beta^+ \subset X).$$

Then $\alpha \in Y$ and $D[Y]$.

Proof. As in [1,4] this is seen from Lemma 6.31. $\Box$

Proposition 6.37 $0 \in X$ *for any distinguished set $X \neq \emptyset$.*

Proof. This is seen from Propositions 6.36 and 6.23.1. $\Box$

The following Lemma 6.38 is a key on distinguished classes.

Lemma 6.38 (Cf. Lemma 3.3.7 in [4].)
Let X be a distinguished set, and suppose for an $\eta < \mathbb{K}$

$$\eta \in \mathcal{G}(X) \cap V(X) \tag{42}$$

and

$$\forall\gamma \prec \eta(\gamma \in \mathcal{G}(X) \cap V(X) \to \gamma \in X) \tag{43}$$

Then

$$\eta \in W(V^*\mathcal{C}^\eta(X)) \cap \eta^+ \text{ and } D[W(V^*\mathcal{C}^\eta(X)) \cap \eta^+].$$

Proof. By Proposition 6.36 and the hypothesis (42) it suffices to show that

$$\forall\beta<\mathbb{K}(X<\beta\,\&\,\beta^+<\eta^+\Rightarrow W(V^*\mathcal{C}^\beta(X))\cap\beta^+\subset X).$$

Assume $X<\beta<\mathbb{K}$ and $\beta^+<\eta^+$. We have to show $W(V^*\mathcal{C}^\beta(X))\cap\beta^+\subset X$. We prove this by induction on $\gamma\in W(V^*\mathcal{C}^\beta(X))\cap\beta^+$. Suppose $\gamma\in V^*\mathcal{C}^\beta(X)\cap\beta^+$ and

$$\mathrm{MIH}: V^*\mathcal{C}^\beta(X)\cap\gamma\subset X$$

We show $\gamma\in X$.

First note that $\gamma\leq X\Rightarrow\gamma\in X$ since if $\gamma\leq\delta$ for some $\delta\in X$, then by $X<\beta$ and $\gamma\in V^*\mathcal{C}^\beta(X)$ we have $\delta<\beta$, $\gamma\in V^*\mathcal{C}^\delta(X)$ and $\delta\in W(V^*\mathcal{C}^\delta(X))\cap\delta^+=X\cap\delta^+$. Hence $\gamma\in W(V^*\mathcal{C}^\delta(X))\cap\delta^+\subset X$. Therefore we can assume that

$$X<\gamma \tag{44}$$

We show first

$$\gamma\in\mathcal{G}(X) \tag{45}$$

First $\gamma\in\mathcal{C}^\gamma(X)$ by $\gamma\in\mathcal{C}^\beta(X)\cap\beta^+$ and Proposition 6.17. Second we show the following claim by induction on $\ell\alpha$:

Claim 6.39 $\alpha\in\mathcal{C}^\gamma(X)\cap\gamma\Rightarrow\alpha\in X$.

Proof of Claim 6.39. Assume $\alpha\in\mathcal{C}^\gamma(X)\cap\gamma$. We have $\alpha\in V(X)$ by $\gamma\in V^*(X)$. Also by Proposition 6.31 we have $\mathcal{C}^\alpha(X)\cap\alpha\subset\mathcal{C}^\gamma(X)\cap\gamma\subset V(X)$. Hence $\alpha\in V^*(X)$, and We have $\alpha\in\mathcal{C}^\beta(X)\cap\gamma\Rightarrow\alpha\in X$ by MIH.

We can assume $\gamma^+\leq\beta$ for otherwise we have $\alpha\in V^*\mathcal{C}^\gamma(X)\cap\gamma=V^*\mathcal{C}^\beta(X)\cap\gamma\subset X$ by MIH. In what follows assume $\alpha\not\in X$.

First consider the case $\alpha\not\in\mathcal{E}(\alpha)$. By induction hypothesis on lengths we have $\mathcal{E}(\alpha)\subset X\subset\mathcal{C}^\beta(X)$, and hence $\alpha\in V^*\mathcal{C}^\beta(X)\cap\gamma$. Therefore $\alpha\in X$ by MIH.

In what follows assume $\alpha=\psi_\pi^{\vec{\nu}}(a)$ for some $\pi>\gamma$ such that $\{\pi,a\}\cup K^2(\vec{\nu})\subset\mathcal{C}^\gamma(X)$.

Case 1. $\beta<\pi$: Then $\forall\kappa\leq\beta[G_\kappa(\{\pi,a\}\cup K^2(\vec{\nu}))=G_\kappa(\alpha)<\alpha<\gamma]$ by Proposition 6.8.1. Proposition 6.25.1 with induction hypothesis on lengths yields $\{\pi,a\}\cup K^2(\vec{\nu})\subset\mathcal{C}^\beta(X)$. Hence $\alpha\in V^*\mathcal{C}^\beta(X)\cap\gamma$ by $\pi>\beta$. MIH yields $\alpha\in X$.

Case 2. $\beta\geq\pi$: We have $\alpha<\gamma<\pi\leq\beta$. It suffices to show that $\alpha\leq X$. Then by (44) we have $\alpha\leq\delta\in X$ for some $\delta<\gamma$. $V^*\mathcal{C}^\delta(X)\ni\alpha\leq\delta\in X\cap\delta^+=W(V^*\mathcal{C}^\delta(X))\cap\delta^+$ yields $\alpha\in W(V^*\mathcal{C}^\delta(X))\cap\delta^+\subset X$.

Consider first the case $\gamma\not\in\mathcal{E}(\gamma)$. By Proposition 6.37 and $\gamma<\beta^+<\mathbb{K}$ we can assume that $\gamma\not\in\{0,\mathbb{K}\}$. Then let $\delta=\max S(\gamma)$ denote the largest immediate subterm of γ. Then $\delta\in\mathcal{C}^\gamma(X)\cap\gamma\subset V(X)$ by $\gamma\in V^*\mathcal{C}^\gamma(X)$, and by (44), $X<\gamma\in\mathcal{C}^\beta(X)$ we have $\delta\in\mathcal{C}^\beta(X)\cap\gamma$. Moreover by Lemma 6.31 we have $\mathcal{C}^\delta(X)\cap\delta\subset\mathcal{C}^\gamma(X)\cap\gamma\subset V(X)$, and $\delta\in V^*\mathcal{C}^\beta(X)\cap\gamma$. Hence $\delta\in X$ by MIH. Also by $\Omega_\alpha=\alpha$, we have $\alpha\leq\delta$, i.e., $\alpha\leq X$, and we are done.

Let $\gamma=\psi_\kappa^{\vec{\xi}}(b)$ for some $b,\vec{\xi}$ and $\kappa>\beta$ by (44). We have $\alpha<\gamma<\pi\leq\beta<\kappa$. $\pi\not\in\mathcal{H}_b(\gamma)$ since otherwise by $\pi<\kappa$ we would have $\pi<\gamma$. Then by Proposition 2.17 we have $a\geq b$ and $K^2(\vec{\xi})\cup\{\kappa,b\}\not\subset\mathcal{H}_a(\alpha)$. On the other hand we have $K_\gamma(K^2(\vec{\xi})\cup\{\kappa,b\})<b\leq a$. By Proposition 6.11 pick a $\delta\in F_\gamma(K^2(\vec{\xi})\cup\{\kappa,b\})$ such that $\mathcal{H}_a(\alpha)\not\ni\delta<\gamma$. Also we have $K^2(\vec{\xi})\cup\{\kappa,b\}\subset\mathcal{C}^\beta(X)$. Hence by Proposition 6.27 we obtain $\delta\in\mathcal{C}^\beta(X)\cap\gamma$. Moreover by Lemma 6.31 we have $\mathcal{C}^\delta(X)\cap\delta\subset\mathcal{C}^\gamma(X)\cap\gamma\subset V(X)$, and $\delta\in\mathcal{C}^\beta(X)\cap\gamma\subset\mathcal{C}^\gamma(X)\cap\gamma\subset V(X)$. Hence $\delta\in V^*\mathcal{C}^\beta(X)\cap\gamma$. Therefore $\alpha\leq\delta\in X$ by MIH. We are done.

Thus Claim 6.39 is shown. □

Hence we have (45), $\gamma\in\mathcal{G}(X)\cap V(X)$. We have $\gamma<\beta^+\leq\eta\,\&\,\gamma\in\mathcal{C}^\gamma(X)$. If $\gamma\prec\eta$, then the hypothesis (43) yields $\gamma\in X$. In what follows assume $\gamma\not\prec\eta$.

If $\forall\tau\leq\eta[G_\tau(\gamma)<\gamma]$, then Proposition 6.25.2 yields $\gamma\in\mathcal{C}^\eta(X)\cap\eta\subset X$ by $\eta\in\mathcal{G}(X)$.

Suppose $\exists\tau\leq\eta[G_\tau(\gamma)=\{\gamma\}]$. This means, by $\gamma\not\prec\eta$, that $\gamma\prec\tau$ for a $\tau<\eta$. Let τ denote the maximal such one. We have $\gamma<\tau<\eta$. Proposition 6.16 with $\gamma\in\mathcal{C}^\gamma(X)$ yields $\tau\in\mathcal{C}^\gamma(X)$.

Next we show that

$$\forall\kappa\leq\eta[G_\kappa(\tau)<\gamma] \tag{46}$$

Let $\kappa\leq\eta$. If $\gamma\not\prec\kappa$, then $G_\kappa(\tau)\subset G_\kappa(\gamma)<\gamma$ by Propositions 6.10 and 6.8.1. If $\gamma\prec\kappa$, then by the maximality of τ we have $\kappa\preceq\tau$, and hence $G_\kappa(\tau)=\emptyset$, cf. (38). (46) is shown.

Hence Proposition 6.25.2 yields $\tau\in\mathcal{C}^\eta(X)$, and $\tau\in\mathcal{C}^\eta(X)\cap\eta\subset X$ by $\eta\in\mathcal{G}(X)$. Therefore $X<\gamma<\tau\in X$. This is not the case by (44). We are done. □

6.4 Mahlo universes

Definition 6.40 *1.* By a *universe* we mean either a *whole universe* L or a transitive set $Q\in$ L in a whole universe L such that $\omega\in Q$. Universes are denoted $P,Q,\ldots$

2. A universe P is said to be a *limit universe* if P is a limit of admissible sets. *Lmtad* denotes the class of limit universes.

3. For a universe P, $\Delta_0(\Delta_1)$ *in* P denotes the class of predicates which are Δ_0 in some Δ_1 predicates on P.

We see the absoluteness of the predicate $D[X]$ over limit universes.

Proposition 6.41 *Let P be a limit universe and $X\in\mathcal{P}(\omega)\cap P$.*

1. *$W(V^*\mathcal{C}^\alpha(X))$ is Δ_1 and $D[X]$ is $\Delta_0(\Delta_1)$.*
2. *$W(V^*\mathcal{C}^\alpha(X))=\{\alpha:P\models\alpha\in W(V^*\mathcal{C}^\alpha(X))\}$ and $D[X]\Leftrightarrow P\models D[X]$.*

Definition 6.42 For a limit universe P set

$$\mathcal{W}^P = \bigcup\{X \in P : D[X]\} = \bigcup\{X \in P : P \models D[X]\}.$$

Thus $\mathcal{W}^{\mathrm{L}} = \mathcal{W}$ for the whole universe L.

Proposition 6.43 *For any limit universe P, $D[\mathcal{W}^P]$.*

Proposition 6.44 *For limit universes P, Q, $Q \in P \Rightarrow \mathcal{W}^Q \subset \mathcal{W}^P \,\&\, \mathcal{W}^Q \in P$.*

Proposition 6.45 *For any limit universe P*

$$\beta \in \mathcal{C}^\alpha(\mathcal{W}^P) \leftrightarrow \exists X \in P\{D[X] \,\&\, \beta \in \mathcal{C}^\alpha(X)\}.$$

In the following Proposition 6.46 by a Π^1_0-*class* we mean a first-order definable class.

Proposition 6.46 *Let $\mathcal{X}$ be a Π^1_0-class such that $\mathcal{X} \subset Lmtad$. Suppose $P \in rM_2(\mathcal{X})$ and $\alpha \in \mathcal{G}(\mathcal{W}^P)$. Then there exists a universe $Q \in P \cap \mathcal{X}$ such that $\alpha \in \mathcal{G}(\mathcal{W}^Q)$.*

Proof. This is seen as in [4]. □

Lemma 6.38 together with Proposition 6.46 yields the following Corollary 6.47, which is the key in our wellfoundedness proofs by distinguished sets.

Corollary 6.47 (Cf. Lemma 6.1 in [3].)
Let $\mathcal{X}$ be a Π^1_0-class such that $\mathcal{X} \subset Lmtad$. Suppose $P \in rM_2(\mathcal{X})$ and $\eta \in \mathcal{G}(\mathcal{W}^P) \cap V(\mathcal{W}^P) \cap \mathbb{K}$.
Assume that there exists a distinguished set $X_1 \in P$ such that

$$\forall Q \in P \cap \mathcal{X}[X_1 \in Q \Rightarrow \eta \in V(\mathcal{W}^Q)] \tag{47}$$

Further assume that any $Q \in P \cap \mathcal{X}$ with $X_1 \in Q$ enjoys the following condition:

$$\forall \gamma \prec \eta\{\gamma \in \mathcal{G}(\mathcal{W}^Q) \cap V(\mathcal{W}^Q) \Rightarrow \gamma \in \mathcal{W}^Q\} \tag{48}$$

Then $\eta \in \mathcal{W}^P$.

Corollary 6.48 *Suppose $L \in rM_2(rM_2(Lmtad))$ and $S(\eta) \not\ni \eta \in \mathcal{G}(\mathcal{W}) \cap \mathbb{K}$. Then $\eta \in \mathcal{W}$.*

Proof. (47) and $\eta \in V(\mathcal{W})$ holds by the condition (39). Also any set $Q \in rM_2(Lmtad)$ enjoys (48) even if $\eta =_{NF} \Omega_{a+1}$. Specifically we have $\forall \gamma \prec \eta\{\gamma \in \mathcal{G}(\mathcal{W}^Q) \cap V(\mathcal{W}^Q)$. This is seen from Corollary 6.47 since there is no $\delta \prec \gamma$. Hence Corollary 6.47 yields $\eta \in \mathcal{W}$. □

Proposition 6.49 *Suppose $L \in rM_2(rM_2(Lmtad))$. Let $\eta < \Lambda$ and $S(\eta) \subset \mathcal{W}$. Then $\eta \in \mathcal{W}$.*
Specifically

1. $\eta =_{NF} \eta_m + \cdots + \eta_0 \,\&\, \{\eta_i : i \leq m\} \subset \mathcal{W} \Rightarrow \eta \in \mathcal{W}\, (m > 0)$.

2. $\eta =_{NF} \varphi\beta\gamma \,\&\, \{\beta, \gamma\} \subset \mathcal{W} \Rightarrow \eta \in \mathcal{W}$.

3. $\eta =_{NF} \Omega_a \,\&\, a \in \mathcal{W} \Rightarrow \eta \in \mathcal{W}$.

Proof. We can assume that $\eta \notin \mathcal{E}(\eta)$ and $\eta \neq 0$ by Proposition 6.37. We have $S(\eta) \subset \mathcal{C}^{\eta}(\mathcal{W})$ by Proposition 6.21, and hence $\eta \in \mathcal{C}^{\eta}(\mathcal{W})$. By Corollary 6.48 it suffices to show

$$\alpha \in \mathcal{C}^{\eta}(\mathcal{W}) \cap \eta \Rightarrow \alpha \in \mathcal{W} \tag{49}$$

6.49.1. It suffices to show that

$$\eta = \beta \dot{+} \gamma \,\&\, \{\beta, \gamma\} \subset \mathcal{W} \Rightarrow \eta \in \mathcal{W}$$

by induction on $\gamma \in \mathcal{W}$, where $\beta \dot{+} \gamma$ designates the fact that the natural sum $\beta \# \gamma = \beta + \gamma$, and $\beta \dot{+} \gamma$ denotes the sum $\beta + \gamma$. We have $\eta \in \mathcal{C}^{\beta}(\mathcal{W}) = \mathcal{C}^{\eta}(\mathcal{W})$. We show (49). If $\alpha < \beta$, then Proposition 6.33 yields $\alpha \in \mathcal{W}$. Let $\alpha = \beta \dot{+} \delta$ with $\delta < \gamma$. Proposition 6.33 yields $\delta \in \mathcal{W}$. IH yields $\alpha \in \mathcal{W}$.

6.49.2. By main induction on $\beta \in \mathcal{W}$ with subsidiary induction on $\gamma \in \mathcal{W}$ we show $\eta = \varphi\beta\gamma \in \mathcal{W}$. We show (49) by induction on $\ell\alpha$. If $\alpha =_{NF} \alpha_m + \cdots + \alpha_0\, (m > 0)$, then the induction hypothesis on the lengths yields $\{\alpha_i : i \leq m\} \subset \mathcal{W}$. By Proposition 6.49.1 we obtain $\alpha \in \mathcal{W}$.

If $\alpha =_{NF} \Omega_a$, then $\alpha \leq \max\{\beta, \gamma\}$. Proposition 6.33 yields $\alpha \in \mathcal{W}$.

Finally let $\alpha =_{NF} \varphi\beta_1\gamma_1$. The induction hypothesis on the lengths yields $\{\beta_1, \gamma_1\} \subset \mathcal{W}$. If $\beta_1 < \beta$, then MIH yields $\alpha \in \mathcal{W}$. If $\beta_1 = \beta$, then $\gamma_1 < \gamma$, and SIH yields $\alpha \in \mathcal{W}$. If $\beta_1 > \beta$, then $\alpha < \gamma$. Proposition 6.33 yields $\alpha \in \mathcal{W}$.

6.49.3. By induction on $a \in \mathcal{W}$ we show $\eta =_{NF} \Omega_a \in \mathcal{W}$. We show (49) by induction on $\ell\alpha$. If either $\alpha =_{NF} \alpha_m + \cdots + \alpha_0\, (m > 0)$ or $\alpha =_{NF} \varphi\beta\gamma$, then the induction hypothesis on the lengths yields $S(\alpha) \subset \mathcal{W}$. By Propositions 6.49.1 and 6.49.2 we obtain $\alpha \in \mathcal{W}$. Let $\alpha =_{NF} \Omega_b$. Then $b \in \mathcal{W} \cap a$, and IH yields $\alpha \in \mathcal{W}$. □

7 Iterating recursively Mahlo operations

In this section we define a tower relation on ordinal terms. An ordinal term is associated with each tower. This extra datum, which is wrongly absent in [3,4], is utilized to show the persistency (39) of the set $V_N(X)$ defined in Definition 7.22.

Definition 7.1 Let $<_1, <_0$ be two transitive relations on ω.

1. The relation $<_E = E(<_1, <_0)$ is on sequences $\langle (n_i^1, n_i^0) : i < \ell \rangle$ of pairs with $<_1$-decreasing first components ($n_{i+1}^1 <_1 n_i^1$), and is defined by $\langle (n_i^1, n_i^0) : i < \ell_0 \rangle <_E \langle (m_i^1, m_i^0) : i < \ell_1 \rangle$ iff either $\exists k \forall i < k \forall j < 2[n_i^j =$

$m_i^j \,\&\, (n_k^1, n_k^0) <_L (m_k^1, m_k^0)]$ or $\ell_0 < \ell_1 \,\&\, \forall i < \ell_0 \forall j < 2[n_i^j = m_i^j]$, where $<_L = L(<_1, <_0)$ denotes the lexicographic ordering:

$$\langle n_1, n_0\rangle <_L \langle m_1, m_0\rangle :\Leftrightarrow n_1 <_1 m_1 \vee (n_1 = m_1 \wedge n_0 <_0 m_0).$$

Write $\sum_{i<\ell} \Lambda^{n_i^1} n_i^0$ for $\langle (n_i^1, n_i^0) : i < \ell\rangle$.

2. Let $dom(<_E)$ denote the domain of the relation $<_E$:

$$dom(<_E) := \{\sum_{i<\ell} \Lambda^{n_i^1} n_i^0 : \forall i < \ell \dot{-} 1(n_{i+1}^1 <_1 n_i^1) \,\&\, n_i^1, n_i^0, \ell \in \omega\}.$$

Definition 7.2 Let $<_i$ $(2 \leq i \leq N-1)$ be transitive Σ_1-relations on ω. Define a *tower* relation $<_T$ from these as follows. Define inductively relations $<_{E_i}$ $(2 \leq i \leq N-1)$.

1. $<_{E_{N-1}} :\equiv <_{N-1}$.
2. $<_{E_i} :\equiv E(<_{E_{i+1}}, <_i)$ for $2 \leq i \leq N-2$, cf. Definition 7.1.

Then let

$$<_T :\equiv <_{E_2} .$$

Definition 7.3 Let $<_i$ $(2 \leq i \leq N-1)$ be transitive Σ_1-relations on ω, and $\prec$ be another transitive Σ_1-relation. Let

$$\mathcal{S} := \{\langle \alpha, \gamma\rangle : \gamma \preceq \alpha\}$$

for the reflexive closure $\preceq$ of $\prec$.

1. Define a *tower* relation $<_{T,p}$ from these as follows. Define relations $<_{E_i,p}$ inductively: $dom(<_{E_{N-1},p}) = \{\langle \alpha, \gamma\rangle \in \mathcal{S} : \alpha \in dom(<_{N-1})\}$, and for $2 \leq i < N-1$ and $\sum_{n<\ell} \Lambda^{\alpha_n} x_n \in dom(<_{E_i})$

$$\langle \sum_{n<\ell} \Lambda^{\alpha_n} x_n, \gamma\rangle \in dom(<_{E_i,p}) :\Leftrightarrow \forall n < \ell(\langle \alpha_n, \gamma\rangle \in dom(<_{E_{i+1},p}) \,\&\, \gamma \preceq x_n)$$

and

$$\langle \alpha, \gamma\rangle <_{E_i,p} \langle \beta, \eta\rangle :\Leftrightarrow \langle \alpha, \gamma\rangle, \langle \beta, \eta\rangle \in dom(<_{E_i,p}) \,\&\, \alpha <_{E_i} \beta \,\&\, \gamma \preceq \eta$$

where $<_{E_i}$ is defined from $\{<_j\}_{j \geq i}$, cf. Definition 7.2. Then let

$$<_{T,p} :\equiv <_{E_2,p} .$$

2.

$$\langle x, \gamma\rangle <_{i,p} \langle y, \eta\rangle :\Leftrightarrow \langle x, \gamma\rangle, \langle y, \eta\rangle \in \mathcal{S} \,\&\, x <_i y \,\&\, \gamma \preceq \eta$$

3. $<_{E_iW,p}$ denotes the restriction of $<_{E_i,p}$ to the wellfounded parts $W(<_{i,p}$) in the second components hereditarily. Namely define inductively for $dom(<_{E_{N-1}}) = dom(<_{N-1})$, $dom(<_{E_{N-1}W,p}) = dom(<_{E_{N-1}}) \times dom(\prec)$, $\langle\alpha, \gamma\rangle <_{E_{N-1}W,p} \langle\beta, \eta\rangle :\Leftrightarrow \langle\alpha, \gamma\rangle <_{E_{N-1},p} \langle\beta, \eta\rangle$.

   For $i < N-1$ and $\sum_{n<\ell} \Lambda^{\alpha_n} x_n \in dom(<_{E_i})$, $\langle\sum_{n<\ell} \Lambda^{\alpha_n} x_n, \gamma\rangle \in dom(<_{E_iW,p})$ iff

$$\forall n < \ell(\langle\alpha_n, \gamma\rangle \in dom(<_{E_{i+1}W,p}) \,\&\, \langle x_n, \gamma\rangle \in W(<_{i,p}))$$

   And let $<_{TW,p}=<_{E_2W,p}$.

Definition 7.4 *1.* $\alpha \in rM_i(X) :\Leftrightarrow \alpha$ is Π_i-reflecting on X.

2. For a definable relation $\lhd$ and set-theoretic universe P (admissibility suffices) let

$$P \in rM_i(a; \lhd) :\Leftrightarrow P \in \bigcap\{rM_i(rM_i(b; \lhd)) : b \lhd^P a\},$$

   where $b \lhd^P a :\Leftrightarrow P \models b \lhd a$.

   Note that $rM_i(a; \lhd)$ is a Π_{i+1}-class for (set-theoretic) Σ_{i+1} $\lhd$.

3. A relation $\lhd$ on ω is said to be *almost wellfounded* in KPℓ if KPℓ proves the transfinite induction schema $TI(a, \lhd)$ up to *each* $a \in \omega$.

In the following Theorem 7.5, $<_i$ $(2 \leq i \leq N-1)$ and $\prec$ denote arbitrary Σ_1-transitive relations on ω such that a weak theory, e.g., KPℓ proves their transitivities.

Let $<_{E_i,p}$ denote the exponential orderings defined from these, and $<_{TW,p}$ denote the restriction of the tower $<_{T,p}=<_{E_2,p}$ to the wellfounded parts $W(<_{i,p})$ in the second components hereditarily.

For $a \in \omega$ and $\langle\alpha, \eta\rangle \in dom(<_{T,p})$ with $\alpha = \sum_{n<\ell} \Lambda^{\alpha_n} x_n$ define inductively

$$\langle\alpha, \eta\rangle < a :\Leftrightarrow \forall n < \ell(\langle\alpha_n, \eta\rangle < a)$$

with $\langle\alpha, \eta\rangle < a :\Leftrightarrow \alpha <_{N-1} a$ for $\alpha \in dom(<_{N-1})$.

The following Theorem 7.5 is seen as in Theorem 3.4 of [3].

Theorem 7.5 *Assume that the relation $<_{N-1}$ is almost wellfounded in* KPℓ. *Then for* each $a \in \omega$,

$$\mathrm{KP}\Pi_N \vdash \mathrm{L} \in \bigcap\{rM_2(rM_2(\langle\alpha, \eta\rangle; <_{TW,p})) : dom(<_{TW,p}) \ni \langle\alpha, \eta\rangle < a\}$$

where L *denotes the whole universe for* KPΠ_N.

7.1 k-predecessors, relations $\prec_k$

In this subsection ordinal terms are *decorated* with indicators.

As in [4] for $2 \leq k \leq N$ and ordinal terms $\alpha = \psi_\pi^{\vec{\nu}}(a)$ with $\vec{\nu} \neq \vec{0}$, the k-predecessor $pd_k(\alpha)$ is defined without mentioning decorations, i.e., indicators in $\nu_i \in E$, cf. Definitions 3.3.3 and 3.3.4. The k-predecessors are needed for us to embed the relation $\prec$ on OT to an exponential structure induced solely from ordinals $\{m_k(\alpha)\}_k$ (cf. Lemma 7.21), which in turn yields sets $V(X) = V_N(X)$ introduced in subsection 7.3 with the persistency (39) in Definition 6.12. As we saw it in section 6, the persistency is crucial for distinguished sets.

Then it truns out that $\alpha \prec pd_k(\alpha)$ holds and the k-predecessor $pd_k(\alpha)$ is determined solely from the sequences $\{\{m_k(\beta)\}_{2\leq k\leq N-1} : \alpha \preceq \beta < \mathbb{K}\}$. Therefore it is convenient for us to handle directly the sequence of sequences $\vec{\nu}$ in defining k-predecessors. After that, let us import them to ordinal terms.

Let $\pi_i = pd(\pi_{i+1})$ for $i < n \leq \omega$ with $\pi_0 = \mathbb{K}$. From Definition 3.3 we see that π_1 is defined from $\mathbb{K}$ (and some b, a) by Definition 3.3.14, each π_{i+1} is defined from π_i by Definition 3.3.15 when $1 < i$ and $i \not\equiv 1 \pmod{(N-2)}$, and each π_{i+1} is defined from π_i by Definition 3.3.16 when $1 < i$ and $i \equiv 1 \pmod{(N-2)}$. This motivates the following.

Let L be a number such that $0 < L \equiv 0 \pmod{(N-2)}$, and π a regular ordinal term such that $pd^{(L+1)}(\pi) = \mathbb{K}$. Let $\pi_n = pd^{(n)}(\pi)$ for $n \leq L+1$, and $\vec{\nu}_n = (\nu_{n2}, \ldots, \nu_{n,N-1})$ be the sequence of decorated ordinals $m_k(\vec{\nu}_n) = \nu_{nk} := m_k(\pi_n) \in E$. Put $\vec{\nu} = \vec{\nu}_0$. Let us write $pd^{(m)}(\vec{\nu}_n) = \vec{\nu}_{n+m}$ for $n + m \leq L$. Otherwise put $pd^{(m)}(\vec{\nu}_n) = \vec{0}$.

Then the following conditions are met for any numbers $n \equiv 0 \pmod{(N-2)}$ and $2 \leq k \leq N-2$ with $n < L$.

1. (Cf. Definition 3.3.14) $\vec{\nu}_L = \vec{0} * (\langle b, \mathbb{K}, a\rangle)$ for some $b \leq a < \Lambda$.

2. (Cf. Definition 3.3.15) $\forall i > k(m_i(pd^{(k-1)}(\vec{\nu}_n)) = 0)$, $\forall i < k(m_i(pd^{(k)}(\vec{\nu}_n)) = m_i(pd^{(k-1)}(\vec{\nu}_n)))$ and for some $b \leq a < \Lambda$, $m_k(pd^{(k-1)}(\vec{\nu}_n)) = m_k(pd^{(k)}(\vec{\nu}_n)) + \Lambda^{m_{k+1}(pd^{(k)}(\vec{\nu}_n))}\langle b, \pi_{n+k}, a\rangle$. In particular
$$m_{k+1}(pd^{(k)}(\vec{\nu}_n)) = te(m_k(pd^{(k-1)}(\vec{\nu}_n))) \tag{50}$$
and
$$m_k(pd^{(N-2)}(\vec{\nu}_n)) = hd(m_k(pd^{(k-1)}(\vec{\nu}_n))) \tag{51}$$

3. (Cf. Definition 3.3.16) $\vec{\nu}_n <_{Ksl} m_2(\vec{\nu}_{n+1})$.

   This means that there exists a sequence $\{p_i(\vec{\nu}_n)\}_{2\leq i\leq N-2}$ of numbers such that, cf. Definition 3.3.7,
$$m_k(\vec{\nu}_n) <_{Kst} hd^{(\vec{p}_k(\vec{\nu}_n))}(m_2(pd(\vec{\nu}_n))) \tag{52}$$
   where $\vec{p}_k(\vec{\nu}_n) = (p_i(\vec{\nu}_n))_{2\leq i\leq k}$.

For ordinals $\nu =_{NF} \Lambda^{\nu_m}\langle b_m, \pi_m, a_m\rangle + \cdots + \Lambda^{\nu_0}\langle b_0, \pi_0, a_0\rangle$, $w(\nu) = m+1$ is the *width* of ν. For sequences $\vec{\nu}$ of ordinals $w_k(\vec{\nu}) := w(m_k(\vec{\nu}))$, the width of the k-th term $m_k(\vec{\nu})$.

Definition 7.6 For $2 \leq k < N$, the *k-predecessor* $pd_k(\vec{\nu})$ of $\vec{\nu}$ is defined recursively.

Define natural numbers $q_k(n, \vec{\nu})\,(2 \leq k \leq N-2, n < w_k(\vec{\nu}))$ and $q_k(\vec{\nu})$ recursively on $k + L$ as follows. $q_k(0, \vec{\nu}) = q_1(\vec{\nu}) = 0$ and

$$q_k(n+1, \vec{\nu}) = q_{k-1}(\vec{\nu}) + (N-2) + q_k(n + p_{k-2}(\vec{\nu}), pd^{(q_{k-1}(\vec{\nu})+(N-2))}(\vec{\nu}))$$

where

$$q_k(\vec{\nu}) = \begin{cases} q_{k-1}(\vec{\nu}) & \text{if } p_k(\vec{\nu}) = 0 \\ r_k(\vec{\nu}) + q_k(pd^{(r_k(\vec{\nu}))}(\vec{\nu})) & \text{if } p_k(\vec{\nu}) > 0 \end{cases} \tag{53}$$

and

$$r_k(\vec{\nu}) = q_{k-1}(\vec{\nu}) + (N-2) + q_k(p_{k-2}(\vec{\nu}) - 1, pd^{(q_{k-1}(\vec{\nu})+(N-2))}(\vec{\nu})).$$

Then put

$$pd_k(\vec{\nu}) = pd^{(q_{k-1}(\vec{\nu})+k-1)}(\vec{\nu}).$$

Proposition 7.7 *Put $r_k(\vec{\nu}) = 0$ if $p_k(\vec{\nu}) = 0$.*

1. $m_k(pd_k(\vec{\nu})) = te(hd^{(\vec{p}_{k-1}(\vec{\nu}))}(m_2(pd(\vec{\nu})))) = te(m_{k-1}(\vec{\nu}))\,(k > 2)$.
2. $n < w_k(\vec{\nu}) \Rightarrow m_k(pd^{(q_k(n,\vec{\nu}))}(\vec{\nu})) = hd^{(n)}(m_k(\vec{\nu}))$.
3. $r_k(\vec{\nu}) > 0 \Rightarrow m_k(pd^{(r_k(\vec{\nu}))}(\vec{\nu})) = hd^{(\vec{p}_k(\vec{\nu}))}(m_2(pd(\vec{\nu})))$.
4. $r_k(\vec{\nu}) > 0 \Rightarrow m_k(\vec{\nu}) <_{Kst} m_k(pd^{(r_k(\vec{\nu}))}(\vec{\nu}))$.
5. $m_k(\vec{\nu}) <_{Kst} m_k(pd^{(q_k(\vec{\nu})+k-1)}(\vec{\nu}))$.

Proof. First we show Propositions 7.7.1, 7.7.2 and 7.7.3 by simultaneous induction on $k + L + n$.
7.7.1. Let $k > 2$. We have $te(hd^{(\vec{p}_{k-1}(\vec{\nu}))}(m_2(pd(\vec{\nu})))) = te(m_{k-1}(\vec{\nu}))$ by (52).

First consider the case $p_{k-1}(\vec{\nu}) = 0$. Then $q_{k-1}(\vec{\nu}) = q_{k-2}(\vec{\nu})$, and for $q = q_{k-2}(\vec{\nu}) + k - 2$, $m_k(pd_k(\vec{\nu})) = m_k(pd^{(q+1)}(\vec{\nu})) = te(m_{k-1}(pd^{(q)}(\vec{\nu}))) = te(m_{k-1}(pd_{k-1}(\vec{\nu})))$ by (50).

When $k = 3$, $q_1(\vec{\nu}) = 0$ and $m_3(pd_3(\vec{\nu})) = te(m_2(pd(\vec{\nu}))) = te(m_2(\vec{\nu}))$ by $p_2(\vec{\nu}) = 0$.

Let $k > 3$. By IH $m_{k-1}(pd_{k-1}(\vec{\nu})) = te(hd^{(\vec{p}_{k-2}(\vec{\nu}))}(m_2(pd(\vec{\nu}))))$, and $m_k(pd_k(\vec{\nu})) = te(te(hd^{(\vec{p}_{k-2}(\vec{\nu}))}(m_2(pd(\vec{\nu}))))) = te(hd^{(\vec{p}_{k-1}(\vec{\nu}))}(m_2(pd(\vec{\nu}))))$ by $p_{k-1}(\vec{\nu}) = 0$.

Next let $p_{k-1}(\vec{\nu}) > 0$. Then $q_{k-1}(\vec{\nu}) = r + q_{k-1}(pd^{(r)}(\vec{\nu}))$ for $r = r_{k-1}(\vec{\nu}) = m + q$ with $m = q_{k-2}(\vec{\nu}) + (N-2)$ and $q = q_{k-1}(p_{k-1}(\vec{\nu}) - 1, pd^{(m)}(\vec{\nu}))$.

By IH we have $m_k(pd_k(pd^{(r)}(\vec{\nu}))) = te(m_{k-1}(pd^{(r)}(\vec{\nu})))$. We obtain by $pd_k(\vec{\nu}) = pd_k(pd^{(r_{k-1}(\vec{\nu}))}(\vec{\nu}))$ and IH on Proposition 7.7.3,
$m_k(pd_k(\vec{\nu})) = m_k(pd_k(pd^{(r)}(\vec{\nu}))) = te(m_{k-1}(pd^{(r)}(\vec{\nu})))$ and $te(m_{k-1}(pd^{(r)}(\vec{\nu}))) =$

$te(hd^{(\vec{p}_{k-1}(\vec{\nu}))}(m_2(pd(\vec{\nu}))))$.

7.7.2. We have $q_k(n+1,\vec{\nu}) = m+q$ for $m = q_{k-1}(\vec{\nu}) + (N-2)$ and $q = q_k(n+p_k(\vec{\nu}), pd^{(m)}(\vec{\nu}))$. On the other hand we have $hd^{(n+p_k(\vec{\nu}))}(m_k(pd^{(m)}(\vec{\nu}))) = m_k(pd^{(q)}(pd^{(m)}(\vec{\nu})))$ by IH. Hence $m_k(pd^{(q_k(n+1,\vec{\nu}))}(\vec{\nu})) = m_k(pd^{(q+m)}(\vec{\nu})) = hd^{(n+p_k(\vec{\nu}))}(m_k(pd^{(m)}(\vec{\nu})))$.

When $k=2$, $m = N-2$ and $m_2(pd^{(N-2)}(\vec{\nu})) = hd(m_2(pd(\vec{\nu})))$ by (51). Hence $m_2(pd^{(q_2(n+1,\vec{\nu}))}(\vec{\nu})) = hd^{(n+1)}(hd^{(p_2(\vec{\nu}))}(m_2(pd(\vec{\nu})))) = hd^{(n+1)}(m_2(\vec{\nu}))$ by (52).

Let $k>2$. By Proposition 7.7.1 $m_k(pd_k(\vec{\nu})) = te(hd^{(\vec{p}_{k-1}(\vec{\nu}))}(m_2(pd(\vec{\nu}))))$. In other words $m_k(pd^{(m)}(\vec{\nu})) = hd(te(hd^{(\vec{p}_{k-1}(\vec{\nu}))}(m_2(pd(\vec{\nu})))))$ by (51). Hence $m_k(pd^{(q_k(n+1,\vec{\nu}))}(\vec{\nu})) = hd^{(n+1)}(hd^{(\vec{p}_k)}(m_2(pd(\vec{\nu})))) = hd^{(n+1)}(m_k(\vec{\nu}))$ by (52).

7.7.3. Let $r_k(\vec{\nu}) = m+q$ for $m = q_{k-1}(\vec{\nu}) + (N-2)$ and $q = q_k(p_k(\vec{\nu}) - 1, pd^{(m)}(\vec{\nu}))$. Then by Proposition 7.7.2 we have
$m_k(pd^{(r_k(\vec{\nu}))}(\vec{\nu})) = m_k(pd^{(q)}(pd^{(m)}(\vec{\nu}))) = hd^{(p_k(\vec{\nu})-1)}(m_k(pd^{(m)}(\vec{\nu})))$.

When $k=2$, we have $m=N-2$ and by (51),
$m_2(pd^{(r_2(\vec{\nu}))}(\vec{\nu})) = hd^{(p_2(\vec{\nu})-1)}(m_2(pd^{(N-2)}(\vec{\nu}))) = hd^{(p_2(\vec{\nu}))}(m_2(pd(\vec{\nu})))$.

Let $k>2$. We have $m_k(pd^{(m)}(\vec{\nu})) = hd(te(hd^{(\vec{p}_{k-1}(\vec{\nu}))}(m_2(pd(\vec{\nu})))))$ by Proposition 7.7.1 and (51). Consequently
$m_k(pd^{(r_k(\vec{\nu}))}(\vec{\nu})) = hd^{(p_k(\vec{\nu}))}(te(hd^{(\vec{p}_{k-1}(\vec{\nu}))}(m_2(pd(\vec{\nu}))))) = hd^{(\vec{p}_k(\vec{\nu}))}(m_2(pd(\vec{\nu})))$.

7.7.4. This is seen from Proposition 7.7.3 and (52).

7.7.5 by induction on $k+L$. First let $p_k(\vec{\nu}) = 0$. Then $q_k(\vec{\nu}) = q_{k-1}(\vec{\nu})$, and $m_k(\vec{\nu}) <_{Kst} te(m_{k-1}(\vec{\nu}))$ by (52) and $te(m_{k-1}(\vec{\nu})) = te(hd^{(\vec{p}_{k-1}(\vec{\nu}))}(m_2(pd(\vec{\nu}))))$. On the other hand we have $m_{k-1}(\vec{\nu}) <_{Kst} m_{k-1}(pd^{(q_{k-1}(\vec{\nu})+k-2)}(\vec{\nu}))$ by IH. Hence $te(m_{k-1}(\vec{\nu})) = te(m_{k-1}(pd^{(q_{k-1}(\vec{\nu})+k-2)}(\vec{\nu}))) = m_k(pd^{(q_{k-1}(\vec{\nu})+k-1)}(\vec{\nu})))$ by (50). Thus $m_k(\vec{\nu}) <_{Kst} m_k(pd^{(q_k(\vec{\nu})+k-1)}(\vec{\nu})))$.

Next let $p_k(\vec{\nu}) > 0$. Then $q_k(\vec{\nu}) = r + q_k(pd^{(r)}(\vec{\nu}))$ for $r = r_k(\vec{\nu})$. Proposition 7.7.4 with IH yields for $q = q_k(pd^{(r)}(\vec{\nu})) + k - 1$,
$m_k(\vec{\nu}) <_{Kst} m_k(pd^{(r)}(\vec{\nu})) <_{Kst} m_k(pd^{(q)}(pd^{(r)}(\vec{\nu}))) = m_k(pd^{(q_k(\vec{\nu})+k-1)}(\vec{\nu}))$.

□

Definition 7.8 *1.* Next let us define the k-predecessor $pd_k(\vec{\nu}_i)$ for $i \not\equiv 0 \pmod{(N-2)}$ as follows.

Let $N-3 \geq i_0 \equiv i \pmod{(N-2)}$. Then put $pd_k(\vec{\nu}_i) := pd(\vec{\nu}_i) = \vec{\nu}_{i+1}$ for any $k \leq i_0 + 2$, and $pd_k(\vec{\nu}_i) := pd^{(N-2-i_0)}(\vec{\nu}_i) = \vec{\nu}_{i-i_0+N-2}$ for $i_0 + 2 < k < N$.

2. $\vec{\nu}_i \prec_k \vec{\nu}_j$ denotes the transitive closure of the relation $\{(\vec{\nu}_i, \vec{\nu}_j) : \vec{\nu}_j = pd_k(\vec{\nu}_i)\}$, and $\vec{\nu}_i \preceq_k \vec{\nu}_j$ its reflexive closure.

$\vec{\nu} \prec_k \vec{\mu}$ indicates that $Mh_k(\vec{\nu}) \prec_k Mh_k(\vec{\xi})$, cf. Definition 2.11.2, Lemma 2.8 for Definition 7.8.1, and Propositions 2.13 and 7.9.6 for Definition 7.6.

Proposition 7.9 *Let $\vec{\mu}, \vec{\xi}$ be in the sequence $\{\vec{\nu}_n\}_{n \leq L}$ with $\vec{\nu}_0 = \vec{\nu}$.*

1. $\vec{\nu} \prec_k pd^{(m)}(\vec{\nu})$ *for* $m = q_{k-1}(\vec{\nu}) + (N-2)$.

2. $\vec{\nu} \preceq_k pd^{(q_k(n,\vec{\nu}))}(\vec{\nu})$.

3. $\vec{\nu} \prec_k pd^{(r_k(\vec{\nu}))}(\vec{\nu})$.

4. $\vec{\mu} \prec_k pd_{k+1}(\vec{\mu})$.

5. *Let* $\vec{\nu} \prec_k \vec{\xi} \prec_k pd^{(r_k(\vec{\nu}))}(\vec{\nu})$ *with* $\vec{\xi} = \vec{\nu}_i$ *for an* $i \equiv 0 \pmod{(N-2)}$. *Then* $m_k(pd^{(r_k(\vec{\nu}))}(\vec{\nu})) = hd^{(n)}(m_k(\vec{\xi}))$ *for some* $n > 0$.

6. *Assume* $\vec{\mu} \prec_k \vec{\xi} \prec_k pd_{k+1}(\vec{\mu})$. *Then* $pd_{k+1}(\vec{\xi}) \preceq_k pd_{k+1}(\vec{\mu})$, *and if* $pd_k(\vec{\mu}) \neq pd_{k+1}(\vec{\mu}) = pd_{k+1}(\vec{\xi})$, *then* $st_k(\vec{\mu}) < st_k(\vec{\xi})$, *and* $K_\sigma(st_k(\vec{\mu})) \leq K_\sigma(st_k(\vec{\xi}))$, *where* $st_k(\vec{\mu})$ *[*σ*] denotes the first [second] component in* $m_k(\vec{\mu}) = \langle st_k(\vec{\mu}), \sigma, a\rangle$, *resp.*

Proof. 7.9.1. By the definition we have $pd_k(\vec{\nu}) = pd^{(m-N+k+1)}(\vec{\nu}) \preceq_k pd^{(m)}(\vec{\nu})$.

7.9.2 by induction on L. We have $q_k(n+1, \vec{\nu}) = m + q$ for $m = q_{k-1}(\vec{\nu}) + (N-2)$ and $q = q_k(n + p_k(\vec{\nu}), pd^{(m)}(\vec{\nu}))$. By Proposition 7.9.1 $\vec{\nu} \prec_k pd^{(m)}(\vec{\nu})$. By IH we have $pd^{(m)}(\vec{\nu}) \preceq_k pd^{(q)}(pd^{(m)}(\vec{\nu}))$.

7.9.3. We have $r = r_k(\vec{\nu}) = m + q_k(p_{k-2}(\vec{\nu}) - 1, pd^{(m)}(\vec{\nu}))$ for $m = q_{k-1}(\vec{\nu}) + (N-2)$, and $\vec{\nu} \prec_k pd^{(m)}(\vec{\nu})$ by Proposition 7.9.1. Proposition 7.9.2 for $n = p_{k-2}(\vec{\nu}) - 1$ yields $\vec{\nu} \prec_k pd^{(r_k(\vec{\nu}))}(\vec{\nu})$.

7.9.4 by induction on L. Let $\vec{\mu} = \vec{\nu}_i$. We can assume $i \leq N-3$.

First consider the case $i \neq 0$. When $k+1 \leq i+2$, we have $pd_{k+1}(\vec{\mu}) = pd_k(\vec{\mu}) = pd(\vec{\mu})$. When $k = i+2$, we have $\forall j < N-i-2(pd_k(\vec{\nu}_{i+j}) = \vec{\nu}_{i+j+1})$ and $pd_{k+1}(\vec{\mu}) = \vec{\nu}_{N-2}$. When $k > i+2$, $pd_{k+1}(\vec{\mu}) = pd_k(\vec{\mu}) = \vec{\nu}_{N-2}$.

Next let $i = 0$. When $p_k(\vec{\nu}) = 0$, we have $q_k(\vec{\nu}) = q_{k-1}(\vec{\nu})$, $pd_k(\vec{\nu}) = pd^{(q_{k-1}(\vec{\nu})+k-1)}(\vec{\nu})$ and $pd_{k+1}(\vec{\nu}) = pd^{(q_{k-1}(\vec{\nu})+k)}(\vec{\nu})$. By the definition $pd_k(pd^{(q_{k-1}(\vec{\nu})+k-1)}(\vec{\nu})) = pd^{(q_{k-1}(\vec{\nu})+k)}(\vec{\nu})$.

Next let $p_k(\vec{\nu}) > 0$. Then $q_k(\vec{\nu}) = r + q_k(pd^{(r)}(\vec{\nu}))$ for $r = r_k(\vec{\nu})$, and $pd_{k+1}(\vec{\nu}) = pd_{k+1}(pd^{(r)}(\vec{\nu}))$. By IH we have $pd^{(r)}(\vec{\nu}) \prec_k pd_{k+1}(pd^{(r)}(\vec{\nu}))$. Proposition 7.9.3 yields $\vec{\nu} \prec_k pd_{k+1}(\vec{\nu})$.

7.9.5. We have $r = r_k(\vec{\nu}) = m_0 + q_k(p_k(\vec{\nu}) - 1, pd^{(m)}(\vec{\nu}))$ for $m_0 = q_{k-1}(\vec{\nu}) + (N-2)$. Hence $m \leq i < r$, and $n_0 = p_k(\vec{\nu}) - 1 > 0$.

If $i = m_0$, then we have $m_k(pd^{(r)}(\vec{\nu})) = hd^{(n_0)}(m_k(pd^{(m_0)}(\vec{\nu})))$ by Proposition 7.7.2.

Let $m_0 < i < r$. We have $q_k(n_0, pd^{(m_0)}(\vec{\nu})) = m_1 + q_k(n_1, pd^{(m_0+m_1)}(\vec{\nu}))$ for $n_1 = n_0 - 1 + p_k(pd^{(m_0)}(\vec{\nu}))$ and $m_1 = q_{k-1}(pd^{(m_0)}(\vec{\nu})) + (N-2)$. Then $m_0 + m_1 \leq i < r$. If $i = m_0 + m_1$, then $n_1 > 0$ and by Proposition 7.7.2 we have $m_k(pd^{(r)}(\vec{\nu})) = hd^{(n_1)}(m_k(pd^{(m_0+m_1)}(\vec{\nu})))$.

In this way se see inductively that there exists a $J \geq 0$ such that $\xi = \vec{\nu}_i = pd^{(\sum_{j \leq J} m_j)}(\vec{\nu})$ for $i = \sum_{j \leq J} m_j$, $n_J > 0$ and

$m_k(pd^{(r)}(\vec{\nu})) = hd^{(n_J)}(m_k(pd^{(\sum_{j\leq J} m_j)}(\vec{\nu})))$, where $m_{j+1} = q_{k-1}(pd^{(m_j)}(\vec{\nu})) + (N-2)$ and $n_{j+1} = n_j - 1 + p_k(pd^{(m_j)}(\vec{\nu}))$ with $m_{-1} = 0, n_{-1} = 1$.

7.9.6 by induction on $L-j$ for $\vec{\xi} = \vec{\nu}_j$. By Proposition 6.2 assuming $pd_{k+1}(\vec{\mu}) = pd_{k+1}(\vec{\xi})$, it suffices to show that $m_k(\vec{\mu}) <_{Kst} m_k(\vec{\xi})$ since we have by (5), $K_\alpha(\{\sigma, st_k(\vec{\mu})\}) < st_k(\vec{\mu})$ for $\alpha = \psi_{\sigma^+}(st_k(\vec{\mu}))$ and similarly for $st_k(\vec{\xi})$. Let $\vec{\mu} = \vec{\nu}_i$. We can assume $i \leq N-3$.

First consider the case $i \neq 0$. From $pd_k(\vec{\mu}) \neq pd_{k+1}(\vec{\mu})$ we see that $k = i+2$, $pd_{k+1}(\vec{\mu}) = \vec{\nu}_{N-2}$, and $pd_k(\vec{\mu}) = \vec{\nu}_{i+1}$. On the other side we see that $\vec{\xi} = \vec{\nu}_{N-3}$, $i = N-4$ and $k = N-2$ from $\vec{\mu} \prec_k \vec{\xi}$ and $pd_{k+1}(\vec{\mu}) = pd_{k+1}(\vec{\xi})$. Then $0 = m_{N-2}(\vec{\mu}) <_{Kst} m_{N-2}(\vec{\xi}) \neq 0$. This shows Proposition 7.9.6 for the case.

Let $\vec{\mu} \prec_k \vec{\xi} \prec_k pd_{k+1}(\vec{\mu})$. Then $pd_{k+1}(\vec{\mu}) = \vec{\nu}_{N-2}$, $k = i+2$, $pd_k(\vec{\mu}) = \vec{\nu}_{i+1}$ and $\vec{\xi} = \vec{\nu}_j$ for $j > i$. Thus $\vec{\xi} \prec_{k+1} \vec{\nu}_{N-2}$.

Next let $i = 0$ and $\vec{\mu} = \vec{\nu}$. (Then $pd_k(\vec{\mu}) \neq pd_{k+1}(\vec{\mu})$.) When $p_k(\vec{\nu}) = 0$, we have $pd(pd_k(\vec{\nu})) = pd_{k+1}(pd_k(\vec{\nu})) = pd_{k+1}(\vec{\nu})$. Also $\vec{\xi} = pd_k(\vec{\nu}) = pd^{(q_k(\vec{\nu})+k-1)}(\vec{\nu})$ and $m_k(\vec{\nu}) <_{Kst} m_k(\vec{\xi})$ by Proposition 7.7.5.

Let $p_k(\vec{\nu}) > 0$. Then by Proposition 7.9.3 we have $\vec{\nu} \prec_k pd^{(r)}(\vec{\nu})$ for $r_\nu = r_k(\vec{\nu})$. Also $pd_{k+1}(\vec{\nu}) = pd_{k+1}(pd^{(r_\nu)}(\vec{\nu}))$, and $m_k(\vec{\nu}) <_{Kst} m_k(pd^{(r_\nu)}(\vec{\nu}))$ by Proposition 7.7.4. Hence by IH we can assume that $\vec{\nu} \prec_k \vec{\xi} \prec_k pd^{(r_\nu)}(\vec{\nu})$ and $\vec{\xi} = \vec{\nu}_i$ for an $i \equiv 0 \pmod{(N-2)}$. By Proposition 7.9.5 we have for some $n_0 > 0$

$$m_k(pd^{(r_\nu)}(\vec{\nu})) = hd^{(n_0)}(m_k(\vec{\xi})) \tag{54}$$

It suffices to show by induction on $L-j$ for $\vec{\xi} = \vec{\nu}_j$ that

$$pd_{k+1}(\vec{\xi}) \preceq_k pd^{(r_\nu)}(\vec{\nu}) \tag{55}$$

By IH and $pd_{k+1}(\vec{\xi}) = pd_{k+1}(pd^{(r_\xi)}(\vec{\xi}))$ for $r_\xi = r_k(\vec{\xi})$ it suffices to show that $pd^{(r_\xi)}(\vec{\xi}) \prec_k pd^{(r_\nu)}(\vec{\nu})$. Assume $pd^{(r_\nu)}(\vec{\nu}) \preceq_k pd^{(r_\xi)}(\vec{\xi})$ contrarily. Then $\vec{\xi} \prec_k pd^{(r_\nu)}(\vec{\nu}) \preceq_k pd^{(r_\xi)}(\vec{\xi})$. Then Proposition 7.9.5 yields $m_k(pd^{(r_\xi)}(\vec{\xi})) = hd^{(n_1)}(m_k(pd^{(r_\nu)}(\vec{\nu})))$ for some $n_1 \geq 0$. On the other hand we have by Proposition 7.7.4 $hd(m_k(pd^{(r_\xi)}(\vec{\xi}))) = hd(m_k(\vec{\xi}))$, and hence by (54) $m_k(pd^{(r_\nu)}(\vec{\nu})) = hd^{(n_0)}(m_k(pd^{(r_\xi)}(\vec{\xi})))$. Therefore $m_k(pd^{(r_\xi)}(\vec{\xi})) = hd^{(n_0+n_1)}(m_k(pd^{(r_\xi)}(\vec{\xi})))$ for $n_0 + n_1 > 0$. This is a contradiction. Thus $pd^{(r_\xi)}(\vec{\xi}) \prec_k pd^{(r_\nu)}(\vec{\nu})$, and (55) is shown. □

Definition 7.10 Next for $\vec{\mu}$ in the sequence $\{\vec{\nu}_n\}_{n\leq L}$ with $\vec{\nu}_0 = \vec{\nu}$, we define sequences $\{\vec{\mu}_k^m\}_{m<lh_k(\vec{\mu})}$ in length $lh_k(\vec{\mu})$ as follows.

1. The case when $\neg\exists\vec{\xi}(\vec{\mu} \preceq_k \vec{\xi} \,\&\, pd_k(\vec{\xi}) \neq pd_{k+1}(\vec{\xi}))$: Then put $lh_k(\vec{\mu}) = 1$ and $\vec{\mu}_k^0 := \vec{\nu}_L$.

2. The case when $\exists\vec{\xi}(\vec{\mu} \preceq_k \vec{\xi} \,\&\, pd_k(\vec{\xi}) \neq pd_{k+1}(\vec{\xi}))$: Then $\vec{\mu}_k^0 = \vec{\nu}_i$ where i is the least number such that $\vec{\mu} \preceq_k \vec{\mu}_k^0$ and $pd_k(\vec{\mu}_k^0) \neq pd_{k+1}(\vec{\mu}_k^0)$.

   Suppose that $\vec{\mu}_k^n$ is defined so that $pd_k(\vec{\mu}_k^n) \neq pd_{k+1}(\vec{\mu}_k^n)$.

(a) The case $\exists\vec{\xi}(pd_{k+1}(\vec{\mu}^n_k) \preceq_k \vec{\xi} \,\&\, pd_k(\vec{\xi}) \neq pd_{k+1}(\vec{\xi}))$: Then $\vec{\mu}^{n+1}_k = \vec{\nu}_i$ where i is the least number such that $pd_{k+1}(\vec{\mu}^n_k) \preceq_k \vec{\mu}^{n+1}_k$ and $pd_k(\vec{\mu}^{n+1}_k) \neq pd_{k+1}(\vec{\mu}^{n+1}_k)$.

(b) Otherwise: Then $lh_k(\vec{\mu}) = n+2$ and define $\vec{\mu}^{n+1}_k = \vec{\nu}_L$.

Proposition 7.11 *For $k < N-1$, $\vec{\mu} \preceq_{k+1} \vec{\mu}^0_k$ and $\forall n < lh_k(\vec{\mu}) - 1[\vec{\mu}^n_k \prec_{k+1} \vec{\mu}^{n+1}_k]$.*

Proof. This is seen from the definition of k-predecessors in Definitions 7.6 and 7.8. □

Proposition 7.12 *Let $\vec{\nu} \prec_k \vec{\xi} \prec_k pd_{k+1}(\vec{\nu})$. Then there exists a $\vec{\mu} \in \{\vec{\xi}\} \cup \{\vec{\xi}^m_k : m < lh(\vec{\xi}) - 1\}$ such that $pd_{k+1}(\vec{\nu}) = pd_{k+1}(\vec{\mu})$ and $st_k(\vec{\mu}) > st_k(\vec{\nu})$. Moreover when $\vec{\mu} \notin \{\vec{\xi}^m_k : m < lh(\vec{\xi}) - 1\}$, $\vec{\nu}^1_k = \vec{\xi}^0_k$ holds.*

Proof. By Proposition 7.9.6 we have $pd_{k+1}(\vec{\xi}) \preceq_k pd_{k+1}(\vec{\nu})$.

When $p_k(\vec{\nu}) = 0$, we see from the proof of Proposition 7.9.6 that $\vec{\xi} = pd_k(\vec{\nu})$, $pd_{k+1}(\vec{\xi}) = pd_{k+1}(\vec{\nu})$ and $st_k(\vec{\nu}) <_{st} st_k(\vec{\xi})$. Also $\vec{\nu}^1_k = \vec{\xi}^0_k$ holds in this case.

Let $p_k(\vec{\nu}) > 0$. We show that by induction on $L - i$ with $\vec{\xi} = \vec{\nu}_i$

$$\vec{\nu} \prec_k \vec{\xi} \preceq_k pd^{(r_0)}(\vec{\nu}) \Rightarrow \exists m < lh(\vec{\xi})[pd^{(r_0)}(\vec{\nu}) = \vec{\xi}^m_k] \tag{56}$$

for $r_0 = r_k(\vec{\nu})$.

If $\vec{\xi} = pd^{(r_0)}(\vec{\nu})$, then $\vec{\xi} = \vec{\xi}^0_k$. Let $\vec{\xi} \neq pd^{(r_0)}(\vec{\nu})$. We can assume that $i \equiv 0 \pmod{(N-2)}$ for $\vec{\xi} = \vec{\nu}_i$, and $\vec{\xi} = \vec{\xi}^0_k$. Otherwise let j be the least number such that $i < j \equiv 0 \pmod{(N-2)}$. Then $\vec{\nu}_j = \vec{\xi}^{m_0}_k \preceq_k pd^{(r_0)}(\vec{\nu})$ for an $m_0 \in \{0, 1\}$. By (55) in the proof of Proposition 7.9.6 we have $pd_{k+1}(\vec{\xi}) \preceq_k pd^{(r_0)}(\vec{\nu})$, and $\vec{\xi}^1_k \preceq_k pd^{(r_0)}(\vec{\nu})$. IH yields (56). Let $pd^{(r_0)}(\vec{\nu}) = \vec{\xi}^m_k$. Then $pd_{k+1}(\vec{\nu}) = pd_{k+1}(\vec{\xi}^m_k)$ and $st_k(\vec{\xi}^m_k) > st_k(\vec{\nu})$ by $pd_{k+1}(\vec{\nu}) = pd_{k+1}(pd^{(r_0)}(\vec{\nu}))$ and Proposition 7.7.4.

Now let $\{r_j\}$ be numbers defined recursively $r_{-1} = 0$, $r_{j+1} = r_k(pd^{(r_j)}(\vec{\nu}))$. If there is a $j \geq -1$ such that $pd^{(r_j)}(\vec{\nu}) \prec_k \vec{\xi} \preceq_k pd^{(r_{j+1})}(\vec{\nu})$, then the proposition is shown by (56). Otherwise there exists a j such that $p_k(pd^{(r_j)}(\vec{\nu})) = 0$ and $pd^{(r_j)}(\vec{\nu}) \prec_k \vec{\xi} \prec_k pd_{k+1}(pd^{(r_j)}(\vec{\nu})) = pd_{k+1}(\vec{\nu})$. Then $\vec{\xi} = pd_k(pd^{(r_j)}(\vec{\nu}))$ and $pd(\vec{\xi}) = pd_{k+1}(\vec{\nu})$. thus $pd_{k+1}(\vec{\xi}) = pd_{k+1}(\vec{\nu})$ and $st_k(\vec{\nu}) <_{st} st_k(\vec{\xi})$. Moreover $\vec{\nu}^1_k = \vec{\xi}^0_k$ holds in this case. □

Proposition 7.13 *Assume $\vec{\xi} = pd_k(\vec{\mu})$ for a $k < N-1$. Then one of the following holds:*

Case 7.13.1 *$\vec{\xi} = pd_{k+1}(\vec{\mu})$, $lh_k(\vec{\mu}) = lh_k(\vec{\xi})$, and*
$\forall m < lh_k(\vec{\mu})[\vec{\mu}^m_k = \vec{\xi}^m_k]$.

Case 7.13.2 *$\vec{\mu}^0_k = \vec{\mu}$, $pd_{k+1}(\vec{\xi}) = pd_{k+1}(\vec{\mu})$, $st_k(\vec{\xi}) > st_k(\vec{\mu})$, and for any $m < lh_k(\vec{\xi}) = lh_k(\vec{\mu}) - 1$, $\vec{\xi}^m_k = \vec{\mu}^{1+m}_k$.*

Case 7.13.3 $\vec{\mu}^0_k = \vec{\mu}$, $pd_{k+1}(\vec{\xi}) \prec_k pd_{k+1}(\vec{\mu})$ *and there exists an* $m < lh(\vec{\xi}) - 1$ *such that* $pd_{k+1}(\vec{\mu}) = pd_{k+1}(\vec{\xi}^m_k)$, $st_k(\vec{\xi}^m_k) > st_k(\vec{\mu})$, *and for any* $0 < i < lh_k(\vec{\xi}) - m = lh_k(\vec{\mu})$, $\vec{\xi}^{m+i}_k = \vec{\mu}^i_k$.

Proof. Assume $\vec{\xi} = pd_k(\vec{\mu})$ for a $k < N - 1$.

First consider the case $pd_k(\vec{\mu}) = pd_{k+1}(\vec{\mu})$. Then $\vec{\mu}^0_i = \vec{\xi}^0_i$, and **Case 7.13.1** holds. Second suppose $pd_k(\vec{\mu}) \neq pd_{k+1}(\vec{\mu})$. Then $\vec{\mu}^0_k = \vec{\mu}$ and $\vec{\mu} \prec_k \vec{\xi} = pd_k(\vec{\mu}) \prec_k pd_{k+1}(\vec{\mu})$. By Proposition 7.12, if $pd_{k+1}(\vec{\xi}) = pd_{k+1}(\vec{\mu})$, then **Case 7.13.2** holds. Otherwise we have $pd_{k+1}(\vec{\mu}) = pd_{k+1}(\vec{\xi}^m_k)$ and $st_k(\vec{\xi}^m_k) > st_k(\vec{\mu})$ for an $m < lh(\vec{\xi}) - 1$. Consequently **Case 7.13.3** holds. □

Now let us define the k-predecessor $pd_k(\alpha)$ of ordinal terms $\alpha = \psi^{\vec{\nu}}_{\pi}(a)$ with $\vec{\nu} \neq \vec{0}$.

Definition 7.14 *1.* The case when $\alpha = \psi^{\vec{\nu}}_{\pi}(a)$ is defined in Definition 3.3.14. Then $\pi = \mathbb{K}$ and $\vec{\nu} = \vec{0} * (\langle b, \mathbb{K}, a\rangle)$. Put $pd_k(\alpha) := \mathbb{K}$ for any k.

2. The case when $\alpha = \psi^{\vec{\nu}}_{\pi}(a)$ is defined in Definition 3.3.15.

Let $k \leq N-2$ be the number in (6) such that $\nu_k = m_k(\pi) + \Lambda^{m_{k+1}(\pi)}\langle b, \pi, a\rangle$. Then put $pd_i(\alpha) := \pi$ for any $i \leq k+1$, and $pd_i(\alpha) := pd^{(N-k)}(\alpha)$ for $k+1 < i < N$, cf. Definition 7.8.1. Also $pd_N(\alpha) = \mathbb{K}$.

3. The case when $\alpha = \psi^{\vec{\nu}}_{\pi}(a)$ is defined in Definition 3.3.16.

Then put $pd_N(\alpha) = \mathbb{K}$, and for $2 \leq k \leq N-1$, $pd_k(\alpha) = pd^{(q_{k-1}(\vec{\nu})+k-1)}(\alpha)$, where $q_{k-1}(\vec{\nu})$ is the number defined from the sequences $\{\{m_k(\beta)\}_{2 \leq k \leq N-1} : \alpha \preceq \beta < \mathbb{K}\}$ in (53) of Definition 7.6.

4. $\alpha \prec_k \beta$ denotes the transitive closure of the relation $\{(\alpha, \beta) : \beta = pd_k(\alpha)\}$.

5. $st_{N-1}(\alpha)$ denotes the first component in $m_{N-1}(\alpha)$, and $st_k(\alpha)$ the first component in $st(m_k(\alpha))$ when $k < N - 1$.

Proposition 7.15 *Let* $\sigma = pd_{k+1}(\alpha) \neq pd_k(\alpha)$ *and* $\beta \preceq \alpha = \psi^{\vec{\nu}}_{\pi}(a)$.

The decorated $st(m_k(\alpha)) = \langle st_k(\alpha), pd_{k+1}(\alpha), a\rangle$ *such that* $\pi \preceq \psi^{\vec{\xi}}_{\sigma}(a)$ *for some* $\vec{\xi}$ *if* $m_k(\alpha) \neq 0$.

Proof. First let $\alpha = \psi^{\vec{\nu}}_{\mathbb{K}}(a)$ with $\vec{\nu} = \vec{0} * (\langle b, \mathbb{K}, a\rangle)$ in Definition 3.3.14. Then $st_{N-1}(\alpha) = b$ and $\mathbb{K} = pd_N(\alpha)$.

Second let $\alpha = \psi^{\vec{\nu}}_{\pi}(a)$ in Definition 3.3.15. By the assumption $pd_{k+1}(\alpha) \neq pd_k(\alpha)$ and Definition 7.14, we have $pd_k(\alpha) = \pi \prec pd_{k+1}(\alpha)$. Then $\forall i > k - 1(m_i(\alpha) = 0)$. In particular $m_k(\alpha) = 0$.

Finally let $\alpha = \psi^{\vec{\nu}}_{\pi}(a)$ in Definition 3.3.16. By Proposition 7.7.5 we have $m_k(\vec{\nu}) <_{Kst} m_k(pd^{(q_k(\vec{\nu})+k-1)}(\vec{\nu}))$ for $\vec{\nu} = \{m_i(\alpha)\}_i$, where $pd^{(q_k(\vec{\nu})+k-1)}(\vec{\nu}) = \{m_i(\gamma)\}_i$ for $\alpha \prec \gamma$ with $pd(\gamma) = pd_{k+1}(\alpha) = \sigma$. In particular $st(m_k(\gamma)) = \langle st_k(\gamma), \sigma, a\rangle$ where $\gamma = \psi^{\vec{\mu}}_{\sigma}(a)$ for some $\vec{\mu}$. Since second and third components in the indicators $m_k(\vec{\nu})$ and in $m_k(pd^{(q_k(\vec{\nu})+k-1)}(\vec{\nu}))$ coincide when $m_k(\vec{\nu}) <_{Kst} m_k(pd^{(q_k(\vec{\nu})+k-1)}(\vec{\nu}))$, we see the proposition. □

Lemma 7.16 *Let $\sigma = pd_{k+1}(\alpha) \neq pd_k(\alpha)$ and $\beta \preceq \alpha = \psi_\pi^{\vec{\nu}}(b)$.*

1. *$F_\sigma(st_k(\alpha)) < \beta$.*
2. *If $\alpha \prec_k \gamma$ and $pd_{k+1}(\gamma) = pd_{k+1}(\alpha)$, then $st_k(\alpha) < st_k(\gamma)$ and $K_\sigma(st_k(\alpha)) \leq K_\sigma(st_k(\gamma))$.*

Proof. 7.16.1. Let $\alpha = \psi_\pi^{\vec{\nu}}(b)$. By Proposition 6.9 we have $F_\pi(st_k(\alpha)) < \beta$, and it suffices to show that $F_\sigma(st_k(\alpha)) < \pi$ by Proposition 6.8.3. We can assume $\pi < \sigma$. By Proposition 7.15 $st(m_k(\alpha)) = \langle st_k(\alpha), \sigma, a\rangle$ where $\pi \preceq \psi_\sigma^{\vec{\xi}}(a)$. Hence by (7) in Definition 3.3.16c we have $K_\pi(st_k(\alpha)) < a$, i.e., $st_k(\alpha) \in \mathcal{H}_a(\pi)$ for $\pi \preceq \psi_\sigma^{\vec{\xi}}(a)$. Hence $F_\sigma(st_k(\alpha)) \subset \mathcal{H}_a(\pi) \cap \sigma$. Let $pd^{(i-1)}(\pi) = \pi_{i-1} = \psi_{\pi_i}^{\vec{\nu}_i}(a_i)$ with $\pi = \pi_0$ and $\sigma = \pi_n$. We have $\mathcal{H}_{a_{j+1}}(\pi_j) \cap \pi_{j+1} \subset \pi_j$ and $a_{j-1} < a_j$ with $a = a_n$. We see by induction on $n - j \geq 0$ that $F_\sigma(st_k(\alpha)) < \pi_j$.

7.16.2. This is seen from Propositions 7.9.6 and 7.15. □

Definition 7.17 Next for terms $\alpha = \psi_\pi^{\vec{\nu}}(a)$ we define sequences $\{\alpha_k^m\}_{m<lh_k(\alpha)}$ in length $lh_k(\alpha)$ by referring Definition 7.10 as follows.

1. The case when $\neg\exists\delta(\alpha \preceq_k \delta \,\&\, pd_k(\delta) \neq pd_{k+1}(\delta))$: Then put $lh_k(\alpha) = 1$ and α_k^0 is defined to be the maximal term such that $\alpha \preceq_{k+1} \alpha_k^0$ with $pd(\alpha_k^0) = \Lambda$.
2. The case when $\exists\delta(\alpha \preceq_k \delta \,\&\, pd_k(\delta) \neq pd_{k+1}(\delta))$: Then α_k^0 is defined to be the minimal term such that $\alpha \preceq_k \alpha_k^0 \,\&\, pd_k(\delta) \neq pd_{k+1}(\delta)$.

   Suppose that α_k^n is defined so that $pd_k(\alpha_k^n) \neq pd_{k+1}(\alpha_k^n)$.

   (a) The case $\exists\gamma(pd_{k+1}(\alpha_k^n) \preceq_k \gamma \,\&\, pd_k(\gamma) \neq pd_{k+1}(\gamma))$: Then α_k^{n+1} is defined to be the minimal term such that $pd_{k+1}(\alpha_k^n) \preceq_k \alpha_k^{n+1}$ and $pd_k(\alpha_{k+1}^n) \neq pd_{k+1}(\alpha_{k+1}^n)$.

   (b) Otherwise: $lh_k(\alpha) = n + 2$ and define α_k^{n+1} to be the maximal term such that $\alpha_k^n \preceq_{k+1} \alpha_k^{n+1}$ with $pd(\alpha_k^0) = \mathbb{K}$.

From Propositions 7.11 and 7.13 we see the following Proposition 7.18 and Lemma 7.19.

Proposition 7.18 *For $i < N - 1$, $\alpha \preceq_{k+1} \alpha_k^0$ and $\forall n < lh_k(\alpha) - 1[\alpha_k^n \prec_{i+1} \alpha_k^{n+1}]$.*

Lemma 7.19 *Assume $\eta = pd_k(\gamma)$ for a $k < N - 1$. Then one of the following holds:*

Case 7.19.1 *$\eta = pd_k(\gamma) = pd_{k+1}(\gamma)$, $lh_k(\gamma) = lh_k(\eta)$, and $\forall m < lh_k(\gamma)[\gamma_k^m = \eta_k^m]$.*

Case 7.19.2 *$\gamma_k^0 = \gamma$, $pd_{k+1}(\eta) = pd_{k+1}(\gamma)$, $st_k(\eta) > st_k(\gamma)$, and for any $m < lh_k(\eta) = lh_k(\gamma) - 1$, $\eta_k^m = \gamma_k^{1+m}$.*

Case 7.19.3 *$\gamma_k^0 = \gamma$, $pd_{k+1}(\eta) \prec_k pd_{k+1}(\gamma)$ and there exists an $m < lh(\eta) - 1$ such that $pd_{k+1}(\gamma) = pd_{k+1}(\eta_k^m)$, $st_k(\eta_k^m) > st_k(\gamma_k^0)$, and for any $0 < i < lh_k(\eta) - m = lh_k(\gamma)$, $\eta_k^{m+i} = \gamma_k^i$.*

7.2 Towers derived from ordinal terms

In this subsection we introduce towers $T(\eta)$ of ordinal terms from the sequence $\{\eta_k^m : m < lh_k(\eta)\}$ defined in Definition 7.17. We will see that the relation $\prec_k$ is embedded in an exponential relation $<_{E_{k,p}}$, cf. Lemma 7.21.

Definition 7.20 *1.* Define relations $<_i$ on OT_n for $2 \leq i \leq N-1$ by

$$\eta <_i \rho :\Leftrightarrow \eta \prec_i \rho \,\&\, pd_i(\eta) \neq pd_{i+1}(\eta) = pd_{i+1}(\rho)$$

2. Extend $<_i$ to $<_i^+$ by adding the successor function $+1$. Namely the domain is expanded to $dom(<_i^+) := dom(<_i) \cup \{a+1 : a \in dom(<_i)\}$, and define for $a, b \in dom(<_i)$, $a+1 <_i^+ b+1 :\Leftrightarrow a <_i b$, $a+1 <_i^+ b :\Leftrightarrow a <_i b$, and $a <_i^+ b+1 :\Leftrightarrow a <_i b$ or $a = b$.

Λ^α denotes $\Lambda^\alpha \cdot 1$.

3. The exponential relations $<_{E_i}, <_{E_i,p}$ are defined from $<_i^+$ $(2 \leq i \leq N-1)$, cf. Definitions 7.2 and 7.3.

4. From the sequence $\{\eta_i^m : 2 \leq i < N-1, m < lh_i(\eta)\}$ we define a tower $T(\eta) = E_2(\eta)$. The elements of the form $E_i(\eta)$ are understood to be ordered by $<_{E_i}$. Let $<_T :\equiv <_{E_2}$.

$$\begin{array}{rcl} E_{N-1}(\eta) & := & \eta \\ E_i(\eta) & := & \displaystyle\sum_{1 \leq m < lh_i(\eta)} \Lambda^{E_{i+1}(\eta_i^m)} \eta_i^{m-1} + \Lambda^{E_{i+1}(\eta_i^0)+1} + \Lambda^{E_{i+1}(\eta)} \end{array}$$

The sequence $\{\eta_i^m : m < lh_i(\eta)\}$ is defined so that the following holds.

Lemma 7.21 *Suppose $\gamma \prec_k \eta$. Then $\langle E_k(\gamma), \gamma\rangle <_{E_k,p} \langle E_k(\eta), \eta\rangle$.*

In particular

$$\gamma \prec_2 \eta \Rightarrow \langle T(\gamma), \gamma\rangle <_{T,p} \langle T(\eta), \eta\rangle$$

Proof by induction on $N-k$.

Let $\gamma \prec_k \eta$. It suffices to show that $E_k(\gamma) <_{E_k} E_k(\eta)$.

$$E_k(\eta) = \sum_{1 \leq n < lh_k(\eta)} \Lambda^{E_{k+1}(\eta_k^n)} \eta_k^{n-1} + \Lambda^{E_{k+1}(\eta_k^0)+1} + \Lambda^{E_{k+1}(\eta)}$$

We can assume $\eta = pd_k(\gamma)$. By Lemma 7.19 one of the following cases occurs.

Case 7.19.1 $\eta = pd_k(\gamma) = pd_{k+1}(\gamma)$, $lh_k(\gamma) = lh_k(\eta)$, and $\forall n < lh_k(\gamma)[\gamma_k^n = \eta_k^n]$. Then

$$E_k(\gamma) = \sum_{1 \leq n < lh_k(\eta)} \Lambda^{E_{k+1}(\eta_k^n)} \eta_k^{n-1} + \Lambda^{E_{k+1}(\eta_k^0)+1} + \Lambda^{E_{k+1}(\gamma)}$$

Case 7.19.2 $\gamma_k^0 = \gamma$, $pd_{k+1}(\eta) = pd_{k+1}(\gamma)$, $st_k(\eta) > st_k(\gamma)$, and for any $n < lh_k(\eta) = lh_k(\gamma) - 1$, $\eta_k^n = \gamma_k^{1+n}$.

$$E_k(\gamma) = \sum_{1 \leq n < lh_k(\eta)} \Lambda^{E_{k+1}(\eta_k^n)} \eta_k^{n-1} + \Lambda^{E_{k+1}(\eta_i^0)} \gamma_k^0 + \Lambda^{E_{k+1}(\gamma_k^0)+1} + \Lambda^{E_{k+1}(\gamma)}$$

Case 7.19.3 $\gamma_k^0 = \gamma$, $pd_{k+1}(\eta) \prec_k pd_{k+1}(\gamma)$ and there exists an $m < lh(\eta) - 1$ such that $pd_{k+1}(\gamma) = pd_{k+1}(\eta_k^m)$, $st_k(\eta_k^m) > st_k(\gamma_k^0)$, and for any $0 < i < lh_k(\eta) - m = lh_k(\gamma)$, $\eta_k^{m+i} = \gamma_k^i$.

$$\begin{aligned}
E_k(\eta) &= \sum_{2 \leq n < lh_k(\gamma)} \Lambda^{E_{k+1}(\gamma_k^n)} \gamma_k^{n-1} + \Lambda^{E_{k+1}(\gamma_k^1)} \eta_k^m + E \\
& \quad (E = \sum_{m \leq n < lh_k(\eta)} \Lambda^{E_{k+1}(\eta_k^n)} \eta_k^{n-1} + \Lambda^{E_{k+1}(\eta_k^0)+1} + \Lambda^{E_{k+1}(\eta)}) \\
E_k(\gamma) &= \sum_{2 \leq n < lh_k(\gamma)} \Lambda^{E_{k+1}(\gamma_k^n)} \gamma_k^{n-1} + \Lambda^{E_{k+1}(\gamma_k^1)} \gamma_k^0 + \Lambda^{E_{k+1}(\gamma_k^0)+1} + \Lambda^{E_{k+1}(\gamma)}
\end{aligned}$$

□

7.3 The sets $V_N(X)$

In this subsection sets $V(X) = V_N(X)$ are defined. Recall that $\mathcal{S} = \{\langle \beta, \alpha \rangle : \alpha \preceq \beta\}$.

Definition 7.22 *1.* For $2 \leq i \leq N-1$,

$$\beta \in U_i(X) :\Leftrightarrow [pd_i(\beta) \neq pd_{i+1}(\beta) \Rightarrow F_{pd_{i+1}(\beta)}(st_i(\beta)) \subset X].$$

And

$$\alpha <_i^X \beta :\Leftrightarrow \alpha, \beta \in U_i(X) \,\&\, \alpha <_i \beta.$$

As in Definition 7.20.2 extend $<_i^X$ to $<_i^{X+}$ by adding the successor function $+1$.

$$\langle \alpha, \alpha_1 \rangle <_{i,p}^X \langle \beta, \beta_1 \rangle :\Leftrightarrow \alpha, \beta \in U_i(X) \,\&\, \langle \alpha, \alpha_1 \rangle <_{i,p} \langle \beta, \beta_1 \rangle$$

for the relation $<_{i,p}$ defined in Definition 7.3.2. The *domain* of $<_{i,p}^X$ is defined to be $\{\langle \alpha, \alpha_1 \rangle \in \mathcal{S} : \alpha \in U_i(X)\}$.

2. For $2 \leq i < N-1$, a finite set $\mathcal{S}_i(\eta)$ of subterms of η is defines as follows:

(a) $\mathcal{S}_2(\eta) := \{\eta_2^m : m < lh_2(\eta)\}$.

(b) For $i > 2$, $\mathcal{S}_i(\eta) := \{\rho_i^m : m < lh_i(\rho), \rho \in \mathcal{S}_{i-1}(\eta)\}$.

Also put $\mathcal{S}_i(\eta) = \emptyset$ if η is not of the form $\psi_\pi^{\vec{\nu}}(a)$.

3. $\eta\in V_N(X)$ designates that each finite set $\mathcal{S}_i(\eta)\times\{\eta\}$ is included in the wellfounded parts $W(<_{i,p}^{X\cap\eta})$ of the relations $<_{i,p}^{X\cap\eta}$.

$$\eta\in V_N(X) :\Leftrightarrow \forall i\in[2,N-1)\forall\beta\in\mathcal{S}_i(\eta)[\beta\in U_i(X)\,\&\,\langle\beta,\eta\rangle\in W(<_{i,p}^{X\cap\eta})].$$

It is clear that $(\bigcup\mathcal{S}_i(\eta))\times\{\eta\}\subset\mathcal{S}$ for any η, and $V_N(X)$ is Δ_1. Suppose $X\cap\alpha_1=Y\cap\alpha_1$ and $\beta\in\mathcal{S}_i(\eta)$ for $\eta\leq\alpha_1$. Then $\eta\preceq\beta$ and $F_{pd_{i+1}(\beta)}(st_i(\beta))<\eta$ by Lemma 7.16.1. Hence $\beta\in U_i(X)$ iff $\beta\in U_i(Y)$. Obviously $\langle\alpha,\gamma\rangle<_i^{X\cap\eta}\langle\beta,\eta\rangle\Leftrightarrow\langle\alpha,\gamma\rangle<_i^{Y\cap\eta}\langle\beta,\eta\rangle$ since $F_{pd_{i+1}(\alpha)}(st_i(\alpha))<\gamma\leq\eta$ by Lemma 7.16.1 and $\gamma\preceq\alpha$, $\gamma\prec\eta$. Therefore $\langle\beta,\eta\rangle\in W(<_{i,p}^{X\cap\eta})$ iff $\langle\beta,\eta\rangle\in W(<_{i,p}^{Y\cap\eta})$. Thus $V_N(X)$ enjoys the condition (39).

Proposition 7.23 *For any limit universe P, if $\gamma\in\mathcal{G}(\mathcal{W}^P)$, then $\forall i\in[2,N-1)[\mathcal{S}_i(\gamma)\subset U_i(\mathcal{W}^P)]$ and $\mathcal{S}_{N-2}(\gamma)\subset U_{N-1}(\mathcal{W}^P)$.*

Proof. Assume $\gamma\in\mathcal{G}(\mathcal{W}^P)$. Let $\delta\in\mathcal{S}_i(\gamma)$, $\nu=st_i(\delta)$ and $\sigma=pd_{i+1}(\delta)$. Then $\gamma\preceq\delta$. We have to show $F_\sigma(\nu)\subset\mathcal{W}^P$. By Lemma 7.16.1 we have $F_\sigma(\nu)<\gamma$.

On the other hand we have $\gamma\in\mathcal{C}^\gamma(\mathcal{W}^P)$, and this yields $\nu\in\mathcal{C}^\gamma(\mathcal{W}^P)$ by the definition of the set $\mathcal{C}^\gamma(\mathcal{W}^P)$. Therefore $F_\sigma(\nu)\subset\mathcal{C}^\gamma(\mathcal{W}^P)$ follows from Proposition 6.27. Thus we have $F_\sigma(\nu)\subset\mathcal{C}^\gamma(\mathcal{W}^P)\cap\gamma\subset\mathcal{W}^P$.

For the case $i=N-2$, let $\mu=st_{N-1}(\delta)$ with $\mathbb{K}=pd_N(\delta)$ and $\delta\in\mathcal{S}_{N-2}(\gamma)$. $F_{\mathbb{K}}(\mu)\subset\mathcal{W}^P\cap\gamma$ is seen from $F_{\mathbb{K}}(\mu)<\gamma$. □

By considering the case $X=\mathcal{W}$, the exponential relations $<_{E_i,p}$ are defined from $<_i^{\mathcal{W}+}$ $(2\leq i\leq N-1)$, cf. Definitions 7.22.1, 7.2 and 7.3. It is clear that each $<_i^{\mathcal{W}+}$ is a transitive Σ_1-relation. Then $<_{\mathcal{W},p}$ denotes the restriction of $<_{T,p}=<_{E_2,p}$ to the wellfounded parts $W(<_{i,p}^{\mathcal{W}})$ in the second components hereditarily. Note that for $\langle\alpha,\gamma\rangle\in dom(<_{T,p})$, $\langle\alpha,\gamma\rangle\in dom(<_{\mathcal{W},p})$ iff $\langle x,\gamma\rangle\in W(<_{i,p}^{\mathcal{W}})$ for each component x occurring in the i-th level of α.

Let $\langle\alpha,\gamma\rangle<_{\mathcal{W},p}^P\langle\beta,\eta\rangle :\Leftrightarrow P\models\langle\alpha,\gamma\rangle<_{\mathcal{W},p}\langle\beta,\eta\rangle$. This means that $\langle\alpha,\gamma\rangle<_{\mathcal{W}^P,p}\langle\beta,\eta\rangle$ for the relation $<_{\mathcal{W}^P,p}$ defined from $<_i^{\mathcal{W}^P+}$.

Lemma 7.24 1. *$<_{N-1}$ is almost wellfounded in* KPℓ.

2. *Let P be any limit universe. Suppose $\eta\in V_N(\mathcal{W}^P)$. Then $\langle T(\eta),\eta\rangle\in dom(<_{\mathcal{W},p}^P)$. Moreover if $\gamma\prec\eta$ and $\gamma\in V_N(\mathcal{W}^P)$, then $\langle T(\gamma),\gamma\rangle<_{\mathcal{W},p}^P\langle T(\eta),\eta\rangle$.*

Proof.
7.24.1. $\gamma<_{N-1}\eta\Leftrightarrow\gamma\prec_{N-1}\eta$, and this implies $st_{N-1}(\gamma)<st_{N-1}(\eta)<\varepsilon_{\mathbb{K}+1}$.

7.24.2. The fact that $\eta\in V_N(\mathcal{W}^P)\Rightarrow\langle T(\eta),\eta\rangle\in dom(<_{\mathcal{W},p}^P)$ is seen from the definition of $<_{\mathcal{W},p}^P$. Assume $\gamma\prec\eta$ and $\gamma\in V_N(\mathcal{W}^P)$. Then by Lemma 7.21 we have $\langle T(\gamma),\gamma\rangle<_{T,p}\langle T(\eta),\eta\rangle$. Moreover we have $\langle T(\gamma),\gamma\rangle\in dom(<_{\mathcal{W},p}^P)$. Hence $\langle T(\gamma),\gamma\rangle<_{\mathcal{W},p}^P\langle T(\eta),\eta\rangle$. □

Lemma 7.25 *If* $P \in rM_2(rM_2(\langle T(\eta), \eta\rangle; <_{\mathcal{W},p}))$*, then* $\eta \in \mathcal{G}(\mathcal{W}^P) \cap V_N(\mathcal{W}^P) \to \eta \in \mathcal{W}^P$.

Proof by induction on $\in$.

Let $\mathcal{X} = rM_2(\langle T(\eta), \eta\rangle; <_{\mathcal{W},p}) \subset Lmtad$. First we show the existence of a distinguished set $X_1 \in P$ such that

$$\forall Q \in P \cap \mathcal{X}[X_1 \in Q \Rightarrow \eta \in V_N(\mathcal{W}^Q)] \tag{47}$$

We have $\forall i \in [2, N-1) \forall \beta \in \mathcal{S}_i(\eta)[F_{pd_{i+1}(\beta)}(st_i(\beta)) \subset \mathcal{W}^P \cap \eta]$. Pick a distinguished set $X_1 \in P$ such that $\forall i \in [2, N-1) \forall \beta \in \mathcal{S}_i(\eta)[F_{pd_{i+1}(\beta)}(st_i(\beta)) \subset X_1 \cap \eta]$. Let $X_1 \in Q \in P \cap \mathcal{X}$. Then $X_1 \subset \mathcal{W}^Q \subset \mathcal{W}^P$, and hence $\forall i \in [2, N-1) \forall \beta \in \mathcal{S}_i(\eta)[F_{pd_{i+1}(\beta)}(st_i(\beta)) \subset \mathcal{W}^Q \cap \eta]$, i.e., $\forall i \in [2, N-1) \forall \beta \in \mathcal{S}_i(\eta)[\beta \in U_i(\mathcal{W}^Q \cap \eta)]$.

Furthermore we have $\langle \beta, \eta\rangle \in W(<_{i,p}^{\mathcal{W}^P \cap \eta})$ and $\mathcal{W}^Q \subset \mathcal{W}^P$ for $\beta \in \mathcal{S}_i(\eta)$. Hence $U_i(\mathcal{W}^Q \cap \eta) \subset U_i(\mathcal{W}^P \cap \eta)$ and $\langle \beta, \eta\rangle \in W(<_{i,p}^{\mathcal{W}^Q \cap \eta})$. We obtain $\eta \in V_N(\mathcal{W}^Q)$.

By Corollary 6.47 it suffices to show (48) for any $Q \in P \cap \mathcal{X}$ such that $X_1 \in Q$.

$$\forall \gamma \prec \eta\{\gamma \in \mathcal{G}(\mathcal{W}^Q) \cap V_N(\mathcal{W}^Q) \Rightarrow \gamma \in \mathcal{W}^Q\} \tag{48}$$

Let $Q \in P \cap \mathcal{X}$, $X_1 \in Q$. Assume that $\gamma \prec \eta$ and $\gamma \in \mathcal{G}(\mathcal{W}^Q) \cap V_N(\mathcal{W}^Q)$. Then $\langle T(\gamma), \gamma\rangle <_{\mathcal{W},p}^Q \langle T(\eta), \eta\rangle$ by Lemma 7.24.2.

Therefore $Q \in rM_2(rM_2(\langle T(\gamma), \gamma\rangle; <_{\mathcal{W},p}))$ by $Q \in \mathcal{X} = rM_2(\langle T(\eta), \eta\rangle; <_{\mathcal{W},p}$). IH on $\in$ yields $\gamma \in \mathcal{W}^Q$. This shows (48). We conclude $\eta \in \mathcal{W}^P$ by Corollary 6.47. □

Lemma 7.26 *For* each $n \in \omega$

$$\mathsf{KP}\Pi_N \vdash \forall \alpha \in OT_n[\alpha \in \mathcal{G}(\mathcal{W}) \cap V_N(\mathcal{W}) \cap \mathbb{K} \to \alpha \in \mathcal{W}].$$

Proof. This is seen from Proposition 6.46, Corollary 7.5 and Lemmas 7.24.1 and 7.25. □

8 Wellfoundedness proof(concluded)

In this section we prove Theorem 1.2, i.e., the wellfoundedness of each initial segment of OT.

Let for $\vec{\xi} = (\xi_2, \ldots, \xi_{N-1})$ with $\xi_i \in E$

$$\begin{aligned} E_n\mathcal{C}^{\mathbb{K}}(\mathcal{W}) &:= \{\xi \in E : K(\xi) \subset \mathcal{C}^{\mathbb{K}}(\mathcal{W}) \cap OT_n\} \\ \vec{E}_n\mathcal{C}^{\mathbb{K}}(\mathcal{W}) &:= \{\vec{\xi} \subset E_n\mathcal{C}^{\mathbb{K}}(\mathcal{W}) : \vec{\xi} \text{ is strongly irreducible}\} \end{aligned}$$

Definition 8.1 For $a \in OT_n$ and strongly irreducible sequences $\vec{\nu} = (\nu_2, \ldots, \nu_{N-1}) \subset E_n$, define:

1.

$$A(a,\vec{\nu}) :\Leftrightarrow \forall\sigma\in\mathcal{W}\cup\{\mathbb{K}\}[\psi^{\vec{\nu}}_{\sigma}(a)\in OT_n \Rightarrow \psi^{\vec{\nu}}_{\sigma}(a)\in\mathcal{W}].$$

2.

$$\mathrm{MIH}(a) :\Leftrightarrow \forall b\in\mathcal{C}^{\mathbb{K}}(\mathcal{W})\cap a\forall\vec{\nu}\in\vec{E}_n\mathcal{C}^{\mathbb{K}}(\mathcal{W})\, A(b,\vec{\nu}).$$

3.

$$\mathrm{SIH}(a,\vec{\nu}) :\Leftrightarrow \forall\vec{\xi}\in\vec{E}_n\mathcal{C}^{\mathbb{K}}(\mathcal{W})[\vec{\xi}<_{lx}\vec{\nu}\Rightarrow A(a,\vec{\xi})].$$

Lemma 8.2 *Assume* $\{a\}\cup K^2(\vec{\xi})\subset\mathcal{C}^{\mathbb{K}}(\mathcal{W})$, $\mathrm{MIH}(a)$, *and* $\mathrm{SIH}(a,\vec{\xi})$ *in Definition 8.1. Then*

$$\forall\kappa\in\mathcal{W}\cup\{\mathbb{K}\}[\psi^{\vec{\xi}}_{\kappa}(a)\in OT_n\Rightarrow\psi^{\vec{\xi}}_{\kappa}(a)\in\mathcal{G}(\mathcal{W})].$$

Proof. Let $\alpha_1=\psi^{\vec{\xi}}_{\kappa}(a)\in OT_n$ with $\kappa\in\mathcal{W}\cup\{\mathbb{K}\}$. We have to show $\alpha_1\in\mathcal{G}(\mathcal{W})$.

By Proposition 6.17.1 we have $\{\kappa,a\}\cup K^2(\vec{\xi})\subset\mathcal{C}^{\mathbb{K}}(\mathcal{W})\subset\mathcal{C}^{\alpha_1}(\mathcal{W})$ and hence by Lemma 6.31

$$\alpha_1\in\mathcal{C}^{\alpha_1}(\mathcal{W})\,\&\,\forall\rho[G_{\rho}(\{\kappa,a\}\cup K^2(\vec{\xi}))\subset\mathcal{W}]$$

Thus it suffices to show the following claim.

Claim 8.3

$$\forall\beta_1\in\mathcal{C}^{\alpha_1}(\mathcal{W})\cap\alpha_1[\beta_1\in\mathcal{W}].$$

Proof of Claim 8.3 by induction on $\ell\beta_1$. Assume $\beta_1\in\mathcal{C}^{\alpha_1}(\mathcal{W})\cap\alpha_1$ and let

$$\mathrm{LIH} :\Leftrightarrow \forall\gamma\in\mathcal{C}^{\alpha_1}(\mathcal{W})\cap\alpha_1[\ell\gamma<\ell\beta_1\Rightarrow\gamma\in\mathcal{W}].$$

We show $\beta_1\in\mathcal{W}$.

Case 0. $\beta_1\not\in\mathcal{E}(\beta_1)$ or $\beta_1\in\mathcal{W}\cap\alpha_1$: Assume $\beta_1\not\in\mathcal{W}$. Then $S(\beta_1)\subset\mathcal{C}^{\alpha_1}(\mathcal{W})\cap\alpha_1$. LIH yields $S(\beta_1)\subset\mathcal{W}$. Hence we conclude $\beta_1\in\mathcal{W}$ from Proposition 6.49.

In what follows consider the cases when $\beta_1=\psi^{\vec{\nu}}_{\pi}(b)$ for some $\pi,b,\vec{\nu}$. We can assume $\{\pi,b\}\cup K^2(\vec{\nu})\subset\mathcal{C}^{\alpha_1}(\mathcal{W})$.

Case 1. $\pi\leq\alpha_1$: Then $\{\beta_1\}=G_{\pi}(\beta_1)\subset\mathcal{W}$ by $\beta_1\in\mathcal{C}^{\alpha_1}(\mathcal{W})$ and Proposition 6.29.

Case 2. $b<a$, $\beta_1<\kappa$ and $K_{\alpha_1}(\{\pi,b\}\cup K(\vec{\nu}))<a$: Let B denote a set of subterms of β_1 defined recursively as follows. First $\{\pi,b\}\cup K^2(\vec{\nu})\subset B$. Let $\alpha_1\leq\beta\in B$. If $\beta=_{NF}\omega^{\gamma}>\mathbb{K}$, then $\gamma\in B$. If $\beta=_{NF}\gamma_m+\cdots+\gamma_0$, then $\{\gamma_i:i\leq m\}\subset B$. If $\beta=_{NF}\varphi\gamma\delta$, then $\{\gamma,\delta\}\subset B$. If $\beta=_{NF}\Omega_{\gamma}$, then $\gamma\in B$. If $\beta=_{NF}\psi^{\vec{\zeta}}_{\sigma}(c)$, then $\{\sigma,c\}\cup K^2(\vec{\zeta})\subset B$.

Then from $\{\pi,b\}\cup K^2(\vec{\nu})\subset\mathcal{C}^{\alpha_1}(\mathcal{W})$ we see inductively that $B\subset\mathcal{C}^{\alpha_1}(\mathcal{W})$. Hence by LIH we have $B\cap\alpha_1\subset\mathcal{W}$. Moreover if $\alpha_1\leq\psi^{\vec{\zeta}}_{\sigma}(c)\in B$, then $c\in K_{\alpha_1}(\{\pi,b\}\cup K(\vec{\nu}))<a$.

We claim that

Claim 8.4 $\forall \beta \in B(\beta \in \mathcal{C}^{\mathbb{K}}(\mathcal{W}))$.

Proof of Claim 8.4 by induction on $\ell\beta$. Let $\beta \in B$. We can assume that $\alpha_1 \leq \beta = \psi_{\sigma}^{\vec{\zeta}}(c)$ by induction hypothesis on the lengths. Then by induction hypothesis we have $\{\sigma, c\} \cup K^2(\vec{\zeta}) \subset \mathcal{C}^{\mathbb{K}}(\mathcal{W})$. On the other hand we have $c < a$. MIH(a) yields $\beta \in \mathcal{W}$. Thus the Claim 8.4 is shown. $\Box$

In particular we obtain $\{\pi, b\} \cup K^2(\vec{\nu}) \subset \mathcal{C}^{\mathbb{K}}(\mathcal{W})$. Moreover we have $b < a$. Therefore once again MIH(a) yields $\beta_1 \in \mathcal{W}$.

Case 3. $b = a$, $\pi = \kappa$, $\forall \delta \in K^2(\vec{\nu})(K_{\alpha_1}(\delta) < a)$ and $\vec{\nu} <_{lx} \vec{\xi}$: As in Claim 8.4 we see that $K^2(\vec{\nu}) \subset \mathcal{C}^{\mathbb{K}}(\mathcal{W})$ from MIH(a). SIH$(a, \vec{\xi})$ yields $\beta_1 \in \mathcal{W}$.

Case 4. $a \leq b \leq K_{\beta_1}(\delta)$ for some $\delta \in K^2(\vec{\xi}) \cup \{\kappa, a\}$: It suffices to find a γ such that $\beta_1 \leq \gamma \in \mathcal{W} \cap \alpha_1$. Then $\beta_1 \in \mathcal{W}$ follows from $\beta_1 \in \mathcal{C}^{\alpha_1}(\mathcal{W})$ and Proposition 6.32.

We see that $a \in K_\delta(\alpha)$ iff $\psi_{\kappa}^{\vec{\xi}}(a) \in k_\delta(\alpha)$ for some $\kappa, \vec{\xi}$, and for each $\psi_{\kappa}^{\vec{\xi}}(a) \in k_\delta(\psi_{\kappa_0}^{\vec{\xi}_0}(a_0))$ there exists a sequence $\{\alpha_i\}_{i \leq m}$ of subterms of $\alpha_0 = \psi_{\kappa_0}^{\vec{\xi}_0}(a_0)$ such that $\alpha_m = \psi_{\kappa}^{\vec{\xi}}(a)$, $\alpha_i = \psi_{\kappa_i}^{\vec{\xi}_i}(a_i)$ for some $\kappa_i, a_i, \vec{\xi}_i$, and for each $i < m$, $\delta \leq \alpha_{i+1} \in \mathcal{E}(C_i)$ for $C_i = \{\kappa_i, a_i\} \cup K^2(\vec{\xi}_i)$.

Pick an $\alpha_2 = \psi_{\kappa_2}^{\vec{\xi}_2}(a_2) \in \mathcal{E}(\delta)$ and an $\alpha_m = \psi_{\kappa_m}^{\vec{\xi}_m}(a_m) \in k_{\beta_1}(\alpha_2)$ for some $\kappa_m, \vec{\xi}_m$ and $a_m \geq b \geq a$. We have $\alpha_2 \in \mathcal{W}$ by $\delta \in \mathcal{C}^{\mathbb{K}}(\mathcal{W})$. We can assume $\alpha_2 \geq \alpha_1$. Then $a_2 \in K_{\alpha_1}(\alpha_2) < a \leq b$, and $m > 2$.

Let $\{\alpha_i\}_{2 \leq i \leq m}$ be the sequence of subterms of α_2 such that $\alpha_i = \psi_{\kappa_i}^{\vec{\xi}_i}(a_i)$ for some $\kappa_i, a_i, \vec{\xi}_i$, and for each $i < m$, $\beta_1 \leq \alpha_{i+1} \in \mathcal{E}(C_i)$ for $C_i = \{\kappa_i, a_i\} \cup K^2(\vec{\xi}_i)$. Let $\{n_j\}_{0 \leq j \leq k}$ $(0 < k \leq m-2)$ be the increasing sequence $n_0 < n_1 < \cdots < n_k \leq m$ defined recursively by $n_0 = 2$, and assuming n_j has been defined so that $n_j < m$ and $\alpha_{n_j} \geq \alpha_1$, n_{j+1} is defined as follows

$$n_{j+1} = \min(\{i : n_j \leq i < m : \alpha_i < \alpha_{n_j}\} \cup \{m\}).$$

If either $n_j = m$ or $\alpha_{n_j} < \alpha_1$, then $k = j$ and n_{j+1} is undefined.

Then we claim that

Claim 8.5 $\forall j \leq k(\alpha_{n_j} \in \mathcal{W}) \,\&\, \alpha_{n_k} < \alpha_1$.

Proof of Claim 8.5. By induction on $j \leq k$ we show first that $\forall j \leq k(\alpha_{n_j} \in \mathcal{W})$. We have $\alpha_{n_0} = \alpha_2 \in \mathcal{W}$. Assume $\alpha_{n_j} \in \mathcal{W}$ and $j < k$. Then $n_j < m$, i.e., $\alpha_{n_{j+1}} < \alpha_{n_j}$, and by $\alpha_{n_j} \in \mathcal{C}^{\alpha_{n_j}}(\mathcal{W})$, we have $C_{n_j} \subset \mathcal{C}^{\alpha_{n_j}}(\mathcal{W})$, and hence $\alpha_{n_j+1} \in \mathcal{E}(C_{n_j}) \subset \mathcal{C}^{\alpha_{n_j}}(\mathcal{W})$. We see inductively that $\alpha_i \in \mathcal{C}^{\alpha_{n_j}}(\mathcal{W})$ for any i with $n_j \leq i \leq n_{j+1}$. Therefore $\alpha_{n_{j+1}} \in \mathcal{C}^{\alpha_{n_j}}(\mathcal{W}) \cap \alpha_{n_j} \subset \mathcal{W}$ by Proposition 6.33.

Next we show that $\alpha_{n_k} < \alpha_1$. We can assume that $n_k = m$. This means that $\forall i(n_{k-1} \leq i < m \Rightarrow \alpha_i \geq \alpha_{n_{k-1}})$. We have $\alpha_2 = \alpha_{n_0} > \alpha_{n_1} > \cdots > \alpha_{n_{k-1}} \geq \alpha_1$, and $\forall i < m(\alpha_i \geq \alpha_1)$. Therefore $\alpha_m \in k_{\alpha_1}(\alpha_2) \subset k_{\alpha_1}(\{\kappa, a\} \cup K^2(\vec{\xi}))$, i.e.,

$a_m \in K_{\alpha_1}(\{\kappa, a\} \cup K^2(\vec{\xi}))$ for $\alpha_m = \psi^{\vec{\xi}_m}_{\kappa_m}(a_m)$. On the other hand we have $K_{\alpha_1}(\{\kappa, a\} \cup K^2(\vec{\xi})) < a$ for $\alpha_1 = \psi^{\vec{\xi}}_{\kappa}(a)$. Thus $a \leq a_m < a$, a contradiction.

The Claim 8.5 is shown, and we obtain $\beta_1 \leq \alpha_{n_k} \in \mathcal{W} \cap \alpha_1$.

This completes a proof of Claim 8.3 and of the lemma. $\Box$

Lemma 8.6 *Suppose* MIH(a) *and* $\kappa \leq \mathbb{K}$. *For any ordinal term* $\beta \in OT_n$

$$F_\kappa(\beta) \subset \mathcal{W} \,\&\, K_\kappa(\beta) < a \Rightarrow \beta \in \mathcal{C}^{\mathbb{K}}(\mathcal{W}).$$

Proof by induction on $\ell\beta$. By IH with Proposition 6.49 we can assume $\beta = \psi^{\vec{\nu}}_{\rho}(b) \geq \kappa$. Then $F_\kappa(\beta) = F_\kappa(\{\rho, b\} \cup K^2(\vec{\nu}))$ and $\{b\} \cup K_\kappa(\{\rho, b\} \cup K^2(\vec{\nu})) = K_\kappa(\beta) < a$. By IH we have $\{\rho, b\} \cup K(\vec{\nu}) \subset \mathcal{C}^{\mathbb{K}}(\mathcal{W})$. MIH$(a)$ with $b < a$ yields $A(b, \vec{\nu})$, and we obtain $\beta = \psi^{\vec{\nu}}_{\rho}(b) \in \mathcal{W}$ by $\rho \in \mathcal{W} \cup \{\mathbb{K}\}$. $\Box$

Proposition 8.7 *For* each $n < \omega$, $\mathsf{KP}\ell \vdash TI[\mathcal{C}^{\mathbb{K}}(\mathcal{W}) \cap \omega_{n+1}(\mathbb{K}+1)]$.

Proof. By metainduction on $n < \omega$ using Proposition 6.24 we see $TI[\mathcal{C}^{\mathbb{K}}(\mathcal{W}) \cap \omega_{n+1}(\mathbb{K}+1)]$, i.e., $Prg[\mathcal{C}^{\mathbb{K}}(\mathcal{W}) \cap \omega_{n+1}(\mathbb{K}+1), \mathcal{Y}] \to \mathcal{C}^{\mathbb{K}}(\mathcal{W}) \cap \omega_{n+1}(\mathbb{K}+1) \subset \mathcal{Y}$ for any definable class $\mathcal{Y}$. $\Box$

Lemma 8.8 *Assume* $\{a\} \cup K^2(\vec{\xi}) \subset \mathcal{C}^{\Lambda}(\mathcal{W})$, MIH$(a)$, *and* SIH$(a, \vec{\xi})$ *in Definition 8.1. Then*

$$\forall \pi \in \mathcal{W} \cup \{\mathbb{K}\}[\psi^{\vec{\xi}}_{\pi}(a) \in OT_n \Rightarrow \psi^{\vec{\xi}}_{\pi}(a) \in V_N(\mathcal{W})].$$

Proof. By Lemmas 8.2 and 7.26 it suffices to show that $\alpha_1 = \psi^{\vec{\xi}}_{\pi}(a) \in V_N(\mathcal{W})$, cf. Definition 7.22. Let $2 \leq i < N-1$, $\beta_1 = \psi^{\vec{\mu}}_{\sigma}(b) \in \mathcal{S}_i(\alpha_1)$. We have to show $\langle \beta_1, \alpha_1 \rangle \in W(<^{\mathcal{W} \cap \alpha_1}_{i,p})$. Suppose $pd_i(\beta_1) \neq pd_{i+1}(\beta_1)$ and $\langle \beta_2, \alpha_2 \rangle <^{\mathcal{W} \cap \alpha_1}_{i,p} \langle \beta_1, \alpha_1 \rangle$. We have $\beta_2 \in U_i(\mathcal{W} \cap \alpha_1)$, and $\langle \beta_2, \alpha_2 \rangle <_{i,p} \langle \beta_1, \alpha_1 \rangle$. Then $\beta_2 \prec_i \beta_1$, $pd_i(\beta_2) \neq pd_{i+1}(\beta_2)$, and $\kappa := pd_{i+1}(\beta_2) = pd_{i+1}(\beta_1)$, cf. Definition 7.20. Hence $\nu := st_i(\beta_2) < st_i(\beta_1)$ by Lemma 7.16.2.

We claim that $\nu \in \mathcal{C}^{\mathbb{K}}(\mathcal{W})$. By Lemma 8.6 it suffices to show that $F_\kappa(\nu) \subset \mathcal{W}$ and $K_\kappa(\nu) < a$. We have $F_\kappa(\nu) \subset \mathcal{W}$ by $\beta_2 \in U_i(\mathcal{W} \cap \alpha_1)$.

By Proposition 6.1.2 we have $b \leq a$. On the other hand we have $K_\kappa(\nu) \leq K_\kappa(st_i(\beta_1))$ by Lemma 7.16.2. Therefore $K_\kappa(\nu) \leq K_\kappa(st_i(\beta_1)) < b \leq a$ by Definitions 3.3.15a and 3.3.16a.

Thus we have shown $\nu = st_i(\gamma_1) \in \mathcal{C}^{\mathbb{K}}(\mathcal{W})$. Therefore $\langle \beta_1, \alpha_1 \rangle \in W(<^{\mathcal{W} \cap \alpha_1}_{i,p})$ is seen by induction on $st_i(\beta_2) \in \mathcal{C}^{\mathbb{K}}(\mathcal{W}) \cap \omega_n(\mathbb{K}+1)$, cf. Proposition 8.7. $\Box$

Proposition 8.9 *For* each $\in \omega$ *and each definable class* $\mathcal{X}$ *of strongly irreducible sequences* $\vec{\xi} = (\xi_2, \ldots, \xi_{N-1})$ *of* $\xi_i < \omega_n(\mathbb{K}+1)$

$$\mathsf{KP}\ell \vdash Prg_{lx}[\vec{E}_n\mathcal{C}^{\mathbb{K}}(\mathcal{W}), \mathcal{X}] \to \forall \vec{\xi} \in \vec{E}_n\mathcal{C}^{\mathbb{K}}(\mathcal{W})(\vec{\xi} \in \mathcal{X})$$

where $Prg_{lx}[\vec{E}_n\mathcal{C}^{\mathbb{K}}(\mathcal{W}), \mathcal{X}]$ *denotes*

$$\forall \vec{\xi} \in \vec{E}_n\mathcal{C}^{\mathbb{K}}(\mathcal{W})[\forall \vec{\nu} \in \vec{E}_n\mathcal{C}^{\mathbb{K}}(\mathcal{W})(\vec{\nu} <_{lx} \vec{\xi} \to \vec{\nu} \in \mathcal{X}) \to \vec{\xi} \in \mathcal{X}].$$

Proof. In Definition 2.14 ordinals $o(\vec{\xi}) < \varepsilon_{\mathbb{K}+2}$ are assigned to strongly irreducible $\vec{\xi}$ so that $\vec{\nu} <_{lx} \vec{\xi} \Rightarrow o(\vec{\nu}) < o(\vec{\xi})$ by Proposition 2.15, and $K(o(\vec{\xi})) \subset \mathcal{C}^{\mathbb{K}}(\mathcal{W})$ if $\vec{\xi} \in \vec{E}_n\mathcal{C}^{\mathbb{K}}(\mathcal{W})$.

Now since $K^2(\vec{\xi}) < \omega_n(\mathbb{K}+1)$, we can replace each occurrence of $\Lambda = \varepsilon_{\mathbb{K}+1}$ in $\vec{\xi}$ by $\lambda_n := \omega_n(\mathbb{K}+1)$: let $o_n(\vec{\nu})$ denote the result of replacing Λ by λ_n in $o(\vec{\nu})$. Then $\vec{\nu} <_{lx} \vec{\xi} \Rightarrow o_n(\vec{\nu}) < o_n(\vec{\xi})$ for any $\vec{\nu}, \vec{\xi}$ such that $K^2(\{\vec{\nu}, \vec{\xi}\}) < \lambda_n$.

Furthermore we have $o_n(\vec{\xi}) < \omega_{n(N-1)}(\mathbb{K}+1)$ since $\mathbb{K}\cdot\omega_n(\mathbb{K}+1) = \omega_n(\mathbb{K}+1)$ for $n > 1$. Hence the proposition follows from Proposition 8.7. $\Box$

Using Lemma 8.8, Propositions 8.7 and 8.9 we see

$$\forall a \in \mathcal{C}^{\mathbb{K}}(\mathcal{W}) \cap \omega_n(\mathbb{K}+1) \forall \vec{\nu} \in \vec{E}_n\mathcal{C}^{\mathbb{K}}(\mathcal{W})\, A(a, \vec{\nu})$$

by main induction on $a \in \mathcal{C}^{\mathbb{K}}(\mathcal{W}) \cap \omega_n(\mathbb{K}+1)$ with subsidiary induction on $\vec{\xi} \in \vec{E}_n\mathcal{C}^{\mathbb{K}}(\mathcal{W})\, (K^2(\vec{\xi}) < \omega_n(\mathbb{K}+1))$ along $<_{lx}$. Hence by induction on $\ell\alpha$ we see that $\alpha \in OT_n \Rightarrow \alpha \in \mathcal{C}^{\mathbb{K}}(\mathcal{W})$. Thus Theorem 6.6, and hence Theorem 1.2 is shown.

8.1 Conservative extensions

For a set Φ of formulas and an ordinal term $\alpha \in OT$, $TI(\alpha, \Phi)$ denotes the schema of transfinite induction up to α:

$$\forall \beta(\forall \gamma < \beta\, \phi(\gamma) \to \phi(\beta)) \to \forall \beta < \alpha\, \phi(\beta)\, (\phi \in \Phi)$$

Π^1_0 denotes the set of set-theoretic formulas in the language $\{\in\}$, $\Pi^1_0(\omega)$ the set of arithmetic formulas in a language of the first-order arithmetic, and EA the elementary recursive arithmetic.

Corollary 8.10
1. *$\mathsf{KP}\Pi_N$ is conservative over $\mathsf{KP}\omega + \{TI(\alpha, \Pi^1_0) : \alpha \in OT \cap \Omega\}$ with respect to $\Sigma_1(\Omega)$-sentences.*
2. *$\mathsf{KP}\Pi_N$ is conservative over $\mathsf{EA} + \{TI(\alpha, \Pi^1_0(\omega)) : \alpha \in OT \cap \Omega\}$ with respect to arithmetic sentences.*
3. *$\mathsf{KP}\Pi_N$ is conservative over $\mathsf{EA} + \{TI(\alpha, \Sigma^0_1(\omega)) : \alpha \in OT \cap \Omega\}$ with respect to Π^0_2-arithmetic sentences. In particular each provably computable function in $\mathsf{KP}\Pi_N$ is defined by α-recursion for an $\alpha \in OT \cap \Omega$.*

Proof. First $\mathsf{KP}\Pi_N \vdash TI(\alpha, \Pi^1_0)$ for each $\alpha \in OT \cap \Omega$ by Theorem 1.2. Second as in [6] we see that proofs in sections 4 and 5 except the proof of Theorem 1.1 in the end of section 5 are formalizable in an intuitionistic fixed point theory $\mathrm{FiX}^i(T_2)$ over the arithmetic theory $T_2 := \mathsf{EA} + \{TI(\alpha, \Pi^1_0(\omega)) : \alpha \in OT \cap \Omega\}$. Namely the relation $(\mathcal{H}_\gamma, \Theta) \vdash^a_b \Gamma$ is a fixed point predicate I of a strictly positive arithmetic formula. Let $\mathsf{KP}\Pi_N \vdash \theta$ for an arithmetic sentence θ. Then as in the end of section 5 we see that $(\mathcal{H}_{\omega_n(\mathbb{K}+1)}, \emptyset) \vdash^\alpha_0 \theta$ for an $n < \omega$ and $\alpha = \psi_\Omega(\omega_n(\mathbb{K}+1))$. Then we see that θ is true by transfinite induction up to

α and applied to an arithmetic formula with the fixed point predicate I. Thus $\mathrm{FiX}^i(T_2) \vdash \theta$.

In $\mathrm{FiX}^i(T_2)$ the fixed point predicate I may occur not only in complete induction schema $\phi(0) \land \forall n(\phi(n) \to \phi(n+1)) \to \forall n\, \phi(n)$, but also in the transfinite induction schema $TI(\alpha, \Phi)$. It is easy to modify the proofs in [5, 8] to show the fact that $\mathrm{FiX}^i(T_2)$ is a conservative extension of T_2. Hence $T_2 \vdash \theta$. Corollaries 8.10.2 and 8.10.3 are shown.

Similarly via an intuitionistic fixed point theory $\mathrm{FiX}^i(T_1)$ over the set theory $T_1 := \mathsf{KP}\omega + \{TI(\alpha, \Pi^1_0) : \alpha \in OT \cap \Omega\}$ we see Corollary 8.10.1. $\square$

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